ON APPROXIMATION SINE WAVE WITH THE 5TH AND 7TH ORDER BEZIER PATHS IN PLANE

by

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There are many studies to approximate to sine curve or sine wave. In this study, it has been examined the way how the sine wave can be written as any order Bezier curve. First, it has been written the 5^{th} and the 7^{th} degree Maclaurin series expansion of the parametric form of sine curve. Also, they are 5^{th} and the 7^{th} order Bezier paths, based on the control points with matrix form in $\mathbf{E}^2$. Hence it has been given the control points of the 5^{th} and the 7^{th} order Bezier curve based on the coefficients of the 5^{th} and the 7^{th} degree Maclaurin series expansion of the sine curves in three steps. Further it has been given the coefficients based on the control points of the 5^{th} and the 7^{th} order Bezier curve too.

Key words: *sine wave, Bezier curve, Maclaurin series, 7^{th} order Bezier curve*

Introduction and preliminaries

A Bezier curve is frequently used in computer graphics and related fields, in vector graphics, used in animation as a tool to control motion, see in [1, 2]. In animation applications such as Adobe Flash and Synfig, Bezier curves are used to outline for example movement. Users outline the wanted path in Bezier curves, and the application creates the needed frames for the object to move along the path. For 3-D animation Bezier curves are often used to define 3-D paths as well as 2-D curves for keyframe interpolation. In [3] a dual unit spherical Bezier-like curve corresponds to a rulled surface by using Study's transference principle and closed ruled surfaces are determined via control points and also, integral invariants of these surfaces are investigated. Researchers have written many publications on Bezier curves, but some of these studies inspired this article. For example: in [4], Bezier curves with curvature and torsion continuity has been examined. In [5, 6], Bezier curves and surfaces has been given. In [7], Bezier curves are designed for computer-aided geometric. Recently equivalence conditions of control points and application to planar Bezier curves have been examined. In [8], Frenet apparatus of the cubic Bezier curves has been examined in $\mathbf{E}^3$. In [9], a cubic trigonometric Bezier-like curve similar to the cubic Bezier curve, with a shape parameter, is presented. In here, first 5^{th} order Bezier curve and its first, second and third derivatives have been examined based on the control points of 5^{th} order Bezier Curve in $\mathbf{E}^3$. We have already examined in cubic Bezier curves and involutes in [8, 10]. The Bertrand and the Mannheim mate of a cubic Bezier curve by using matrix representation have been researched in $\mathbf{E}^3$ [11, 12], respectively. In [13], it has been examined the 5^{th} order Bezier curve and its derivatives. In [14], it has been researched the answer of the question "How to find a n^{th} order Bezier curve if we

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know the first, second and third derivatives? Also in [15] it has been given the way how we can determine the wanted 5th order Bezier curve, if we know its the first, the second, and the third derivatives, which it has the wanted control points. And finally, in [14], approximation of circular arcs and helices have been studied. Generaly nth order Beziers curve can be defined by $n+1$ control points $P_0, P_1, \ldots, P_n$ with the parametrization:

$$\mathbf{B}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} [P_i]$$

We have already known that the matrix representation of $\alpha(t) = (t, a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a t_1 + a_0)$ as 5th order Bezier path in $\mathbf{E^2}$ is:

$$\alpha(t) = [t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^5][P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5]^T$$

where the coefficient matrix and the inverse of the 5th order Bezier curve are:

$$[B^5] = \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad [B^5]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 1 \\ 0 & 0 & 0 & \frac{1}{10} & \frac{2}{5} & 1 \\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & 1 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Also the control points are:

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5]^T = [B^5]^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}^T$$

For more detail see [1, 13].

In this study, it will be focused on the 5th and 7th order Bezier paths in $\mathbf{E^2}$. It is well known that Taylor series of a function is an infinite sum of the functions derivatives at a single point, also a Maclaurin series is a Taylor series where $a = 0$. For any function Taylor series expansion is

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}$$

also a Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$$

is a taylor series where $a = 0$.

Sine wave as a 5th order Bezier path

In this section, it has been focused on three types of sine wave as a 5th order Bezier path. First, it will be examined sine function $f(x)=\sin x$.

Theorem 1. The matrix representation of the sine wave of function $f(x)=\sin x$ as a 5th order Bezier curve is:

$$(t,\sin t)=[t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^5]\begin{bmatrix} 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{5}{5} & 1 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{7}{12} & \frac{11}{15} & \frac{101}{120} \end{bmatrix}^T$$

with the control points:

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5]^T=\begin{bmatrix} 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{5}{5} & 1 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{7}{12} & \frac{11}{15} & \frac{101}{120} \end{bmatrix}^T$$

Proof. For sine function $f(x)=\sin x$, the 5th degree Maclaurin series expansion:

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}$$

can be written as in the parametric form and a 5th degree polynomial function:

$$(t,\sin t)=\left(t,\frac{t^5}{5!}-\frac{t^3}{3!}+t\right)=(t,a_5t^5+a_4t^4+a_3t^3+a_2t^2+a_1t+a_0)$$

Also this can be written in matrix form with the matrix representation of 5th order Bezier path as in:

$$(t,\sin t)=[t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1]^T\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{5!} & 0 & -\frac{1^3}{3!} & 0 & 1 & 0 \end{bmatrix}=$$

$$=[t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^5][P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5]$$

Solving the equation we get the control points P_0, P_1, P_2, P_3, P_4, and 5th.

Secondly, let's examine the sine function $f(x)=a\sin bx$, as any 5th order Bezier path. The control points of the 5th order Bezier path have been determined based on the coefficients *a* and *b*.

Theorem 2. The matrix representation of the sine wave of the function $f(x)=a\sin bx$ as a 5th order Bezier path is:

$$(t,a\sin bt)=[t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^5]\begin{bmatrix} 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \\ 0 & \frac{ab}{5} & \frac{2ab}{5} & \frac{36ab-ab^3}{60} & \frac{12ab-ab^3}{15} & \frac{ab^5-20ab^3+120ab}{120} \end{bmatrix}^T$$

with the control points:

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5]^T = \begin{bmatrix} 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{5}{5} & 1 \\ 0 & \frac{ab}{5} & \frac{2ab}{5} & \frac{36ab-ab^3}{60} & \frac{12ab-ab^3}{15} & \frac{ab^5-20ab^3+120ab}{120} \end{bmatrix}^T$$

Proof. We need to write $f(x)=a\sin bx$ in Maclaurin series expansion. For sine function $f(x)=a\sin bx,$ as any 5th degree Maclaurin series expansion is:

$$f(x)=\sum_{n=0}^{5}(a\sin bx)^{(n)}(0)\frac{x^n}{n!}=abx-\frac{ab^3x^3}{3!}+\frac{ab^5x^5}{5!}$$

This 5th degree polynomial function can be written as in parametric form:

$$(t, a\sin bt)=\left(t, \frac{ab^5}{5!}t^5-\frac{ab^3}{3!}t^3+abt\right)=(t, a_5t^5+a_4t^4+a_3t^3+a_2t^2+a_1t+a_0)$$

Also this can be written in matrix form with the matrix representation of 5th order Bezier path as in:

$$(t, a\sin bt)=\begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{ab^5}{5!} & 0 & -\frac{ab^3}{3!} & 0 & ab & 0 \end{bmatrix}=\begin{bmatrix} t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^5]\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}$$

Hence solving the equation as in the following way we get the proof:

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5]^T=[B^5]^{-1}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{ab^5}{5!} & 0 & -\frac{ab^3}{3!} & 0 & ab & 0 \end{bmatrix}$$

Corollary 1. The apscissas and ordinates of the control points the control points $P_0=(x_0,y_0), P_{10}=(x_1,y_1), P_2=(x_2,y_2), P_3=(x_3,y_3), P_4=(x_4,y_4),$ and $P_5=(x_5,y_5)$ curve $\alpha(t)=(t, a\sin bt)$ have the following representations based on the coefficients. Under the following conditions 5th Bezier curve can be written $f(x)=a\sin bx$ curve in plane:

$$x_0=0 \; x_1=\frac{1}{5} \; x_2=\frac{2}{5} \; x_3=\frac{3}{5} \; x_4=\frac{4}{5} \; x_5=1$$

and

$$[y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5]^T=[B^5]^{-1}\begin{bmatrix} \frac{ab^5}{5!} & 0 & -\frac{ab^3}{3!} & 0 & ab & 0 \end{bmatrix}^T$$

Now it will be examined a more complex sine wave $f(x)=a\sin(bx-c)$ as a 5th order Bezier path.

Theorem 3. The matrix representation of the sine wave $f(x)=a\sin(bx-c)$ as a 5th order Bezier path is: $[t, a\sin(bt-c)]=[t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^5][P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5]^T$ with the control points $P_0, P_1, P_2, P_3, P_4,$ and P_5 are:

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 0 & -a\sin c \\ \frac{1}{5} & \frac{1}{5}ab\cos c - a\sin c \\ \frac{2}{5} & \frac{ab^2}{20}\sin c + \frac{2ab}{5}\cos c - a\sin c \\ \frac{3}{5} & -\frac{ab^3}{60}\cos c + \frac{3ab^2}{20}\sin c + \frac{3ab}{5}\cos c - a\sin c \\ \frac{4}{5} & \frac{ab^4}{120}\sin c - \frac{ab^3}{15}\cos c + \frac{3ab^2}{10}\sin c + \frac{4ab}{5}\cos c - a\sin c \\ 1 & \frac{ab^5}{120}\cos c - \frac{ab^4}{24}\sin c - \frac{ab^3}{6}\cos c + \frac{ab^2}{2}\sin c + ab\cos c - a\sin c \end{bmatrix}$$

Proof. Let's examine the sine wave as a 5th order Bezier path. First, we need to write $f(x)=a\sin(bx-c)$ in Maclaurin series expansion. For sine function 5th degree Maclaurin series expansion is:

$$f(x)=\sum_{n=0}^{5}[a\sin(bx-c)]^{(n)}(0)\frac{x^n}{n!}=$$
$$=a\sin(bx-c)=a\sin(-c)+ab[\cos(-c)](x)+\frac{-ab^2\sin(-c)}{2!}x^2+$$
$$+\frac{-ab^3\cos(-c)}{3!}x^3+\frac{ab^4\sin(-c)}{4!}x^4+\frac{ab^5\cos(-c)}{5!}x^5$$

It can be written as in parametric form and a 5th degree polynomial function:

$$[t, a\sin(bt-c)]=\left[t, \frac{ab^5\cos c}{5!}t^5-\frac{ab^4\sin c}{4!}t^4-\frac{ab^3\cos c}{3!}t^3+\frac{ab^2\sin c}{2!}t^2+ab(\cos c)t-a\sin c\right]$$

Also this can be written in matrix form with the matrix representation of 5th order Bezier path as in:

$$[t, a\sin(bt-c)]=$$
$$=[t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{ab^5\cos c}{5!} & \frac{-ab^4\sin c}{4!} & \frac{-ab^3\cos c}{3!} & \frac{ab^2\sin c}{2!} & \text{abcos}c & -\text{asin}c \end{bmatrix}=$$
$$=[t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^5][P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5]^T$$

Solving the equation we get the control points as in the result of the matrix product:

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5]^T=[B^5]^{-1}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{ab^5\cos c}{5!} & \frac{-ab^4\sin c}{4!} & \frac{-ab^3\cos c}{3!} & \frac{ab^2\sin c}{2!} & ab\cos c & -a\sin c \end{bmatrix}^T$$

Corollary 2. The apscissas and ordinates of the control points of sine wave $\alpha(t)=(t, a\sin bt)$ as a 5th order Bezier path are given in the following way, respectively, where $[B^5]^{-1}$ is the inverse of any 5th order Bezier path matrix

$$x_0=0\; x_1=\frac{1}{5}\; x_2=\frac{2}{5}\; x_3=\frac{3}{5}\; x_4=\frac{4}{5}\; x_5=1$$

and

$$[y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5]^T=[B^5]^{-1}\left[\frac{ab^5\cos c}{5!}\,\frac{-ab^4\sin c}{4!}\,\frac{-ab^3\cos c}{3!}\,\frac{ab^2\sin c}{2!}\,ab\cos c - a\sin c\right]^T$$

Corollary 3. The coefficients of the $[t, a\sin(bt-c)]$ based on the only ordinates of the control points of the 5th order Bezier path are:

$$\begin{aligned}
\frac{1}{120}ab^5\cos c &= 5y_1-y_0-10y_2+10y_3-5y_4+y_5\\
-\frac{1}{24}ab^4\sin c &= 5y_0-20y_1+30y_2-20y_3+5y_4\\
-\frac{1}{6}ab^3\cos c &= 30y_1-10y_0-30y_2+10y_3\\
\frac{1}{2}ab^2\sin c &= 10y_0-20y_1+10y_2\\
ab\cos c &= 5y_1-5y_0\\
-a\sin c &= y_0
\end{aligned}$$

Sine wave as a 7th order Bezier path

In this section it will be focused on how it can be writen the sine wave as a 7th order Bezier path. Hence, we need to coefficients matrix of 7th order Bezier path which is given by the following theorem.

Theorem 4. The coefficients matrix and the inverse matrix of any 7th order Bezier path are (for more detail see [13]):

$$[B^7]=\begin{bmatrix}
-1 & 7 & -21 & 35 & -35 & 21 & -7 & 1\\
7 & -42 & 105 & -140 & 105 & -42 & 7 & 0\\
-21 & 105 & -210 & 210 & -105 & 21 & 0 & 0\\
35 & -140 & 210 & -140 & 35 & 0 & 0 & 0\\
-35 & 105 & -105 & 35 & 0 & 0 & 0 & 0\\
21 & -42 & 21 & 0 & 0 & 0 & 0 & 0\\
-7 & 7 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

and

$$[B^7]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{21} & \frac{2}{7} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{35} & \frac{1}{7} & \frac{3}{7} & 1 \\ 0 & 0 & 0 & \frac{1}{35} & \frac{4}{35} & \frac{2}{7} & \frac{4}{7} & 1 \\ 0 & 0 & \frac{1}{21} & \frac{1}{7} & \frac{2}{7} & \frac{10}{21} & \frac{5}{7} & 1 \\ 0 & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \frac{5}{7} & \frac{6}{7} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

In this section it has been focused on sine wave as a 7th order Bezier path. First we will examine $f(x) = \sin x$.

Theorem 6. The numerical matrix representation of the sine wave $f(x) = \sin x$ as a 7th order Bezier path is:

$$(t, \sin t) = [t^7 \quad t^6 \quad t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^7] \begin{bmatrix} 0 & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \frac{5}{7} & \frac{6}{7} & 1 \\ 0 & \frac{1}{7} & \frac{2}{7} & \frac{89}{210} & \frac{58}{105} & \frac{1681}{2520} & \frac{107}{140} & \frac{4241}{5040} \end{bmatrix}^T$$

where the control points are $P_0, P_1, P_2, P_3, P_4, P_5, P_6,$ and P_7.

Proof. For sine function $f(x) = \sin x,$ the 7th degree Maclaurin series expansion and parametric form is:

$$(t, \sin t) = \left(t, t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \right) = (t, a_7 t^7 + a_6 t^6 + a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a t_1 + a_0)$$

Also this can be written in matrix form with the matrix representation of 7th order Bezier path as in:

$$(t, \sin t) = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{1^7}{7!} & 0 & \frac{1}{5!} & 0 & -\frac{1^3}{3!} & 0 & 1 & 0 \end{bmatrix}^T = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^7] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix}$$

Solving the equation we get the control points $P_0, P_1, P_2, P_3, P_4, P_5, P_6,$ and P_7.

Theorem 7. The matrix representation of the sine wave of function $f(x) = a\sin bx$ as any 7th order Bezier path based on the coefficients is:

$$(t, a\sin bt) = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^7] \begin{bmatrix} 0 & 0 \\ \frac{1}{7} & \frac{1}{7}ab \\ \frac{2}{7} & \frac{2}{7}ab \\ \frac{3}{7} & \frac{3}{7}ab - \frac{1}{210}ab^3 \\ \frac{4}{7} & \frac{4}{7}ab - \frac{2}{105}ab^3 \\ \frac{5}{7} & \frac{1}{2520}ab^5 - \frac{1}{21}ab^3 + \frac{5}{7}ab \\ \frac{6}{7} & \frac{1}{420}ab^5 - \frac{2}{21}ab^3 + \frac{6}{7}ab \\ 1 & -\frac{1}{5040}ab^7 + \frac{1}{120}ab^5 - \frac{1}{6}ab^3 + ab \end{bmatrix}$$

with the control points $P_0, P_1, P_2, P_3, P_4, P_5, P_6,$ and P_7 have as the apscissas:

$$x_0 = 0,\ x_1 = \frac{1}{7},\ x_2 = \frac{2}{7},\ x_3 = 0,\ x_4 = \frac{2}{7},\ x_5 = \frac{2}{7},\ x_6 = \frac{2}{7},\ x_7 = 1$$

and the ordinates:

$$y_0 = 0\ y_1 = \frac{1}{7}ab\ y_2 = \frac{2}{7}ab\ y_3 = \frac{3}{7}ab - \frac{1}{210}ab^3\ y_4 = \frac{4}{7}ab - \frac{2}{105}ab^3$$

$$y_5 = \frac{1}{2520}ab^5 - \frac{1}{21}ab^3 + \frac{5}{7}ab\ y_6 = \frac{1}{420}ab^5 - \frac{2}{21}ab^3 + \frac{6}{7}ab\ y_7 = $$

$$= -\frac{1}{5040}ab^7 + \frac{1}{120}ab^5 - \frac{1}{6}ab^3 + ab$$

Proof. For sine function $f(x) = a\sin bx$, the 7th degree Maclaurin series expansion is:

$$f(x) = \sum_{n=0}^{7} (a\sin bx)^{(n)}(0)\frac{x^n}{n!} = abx - \frac{ab^3x^3}{3!} + \frac{ab^5x^5}{5!} - \frac{ab^7x^7}{7!}$$

This 7th degree polynomial function can be written as in parametric form:

$$(t, a\sin bt) = \left(t, \frac{ab^5}{5!}t^5 - \frac{ab^3}{3!}t^3 + abt\right) = (t, a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0)$$

Also this can be written in matrix form with the matrix representation of 7th order Bezier path as in:

$$(t, a\sin bt) = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\dfrac{\mathrm{ab}^7}{7!} & 0 & \dfrac{\mathrm{ab}^5}{5!} & 0 & -\dfrac{ab^3}{3!} & 0 & \mathrm{ab} & 0 \end{bmatrix} = \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^7] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix}$$

Solving the equation we get the control points $P_0, P_1, P_2, P_3, P_4, P_5, P_6,$ and P_7 as in the result of the matrix product:

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5 \quad P_6 \quad P_7]^T = [B^7]^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\dfrac{ab^7}{7!} & 0 & \dfrac{ab^5}{5!} & 0 & -\dfrac{ab^3}{3!} & 0 & ab & 0 \end{bmatrix}^T$$

Theorem 8. The matrix representation of the sine wave of $f(x) = a\sin(bx - c)$ as a 7th order Bezier path is:

$$[t, a\sin(bt - c)] = [t^7 \quad t^6 \quad t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^7][P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5 \quad P_6 \quad P_7]^T$$

where the control points $P_0, P_1, P_2, P_3, P_4, P_5, P_6,$ and P_7 are:

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix} \begin{bmatrix} 0 & -a\sin c \\ \dfrac{1}{7} & \dfrac{1}{7}ab\cos c - a\sin c \\ \dfrac{2}{7} & \dfrac{ab^2}{210}\cos c + \dfrac{1ab^2}{14}\sin c + \dfrac{3ab}{7}\cos c - a\sin c \\ \dfrac{3}{7} & --\dfrac{ab^3}{210}\cos c - \dfrac{1ab^2}{14}\sin c + \dfrac{3ab}{7}\cos c - a\sin c \\ \dfrac{4}{7} & -\dfrac{ab^4\sin c}{840} - \dfrac{2ab^3\cos c}{105} + \dfrac{ab^2\sin c}{7} + \dfrac{4ab\cos c}{7} - a\sin c \\ \dfrac{5}{7} & \dfrac{ab^5\cos c}{2520} - \dfrac{ab^4\sin c}{168} - \dfrac{ab^3\cos c}{21} + \dfrac{5ab^2\sin c}{21} + \dfrac{5ab\cos c}{7} - a\sin c \\ \dfrac{6}{7} & \dfrac{ab^6}{5040}\sin c + \dfrac{ab^5\cos c}{420} - \dfrac{ab^4\sin c}{56} - \dfrac{2ab^3\cos c}{21} + \dfrac{5ab^2\sin c}{14} + \dfrac{6ab\cos c}{7}a\sin c \\ 1 & \dfrac{-ab^7}{5040}\cos c + \dfrac{ab^6}{720}\sin c\dfrac{ab^5\cos c}{120} - \dfrac{ab^4\sin c}{24} - \dfrac{ab^3\cos c}{6} + \dfrac{ab^2\sin c}{2} + ab\cos c - a\sin c \end{bmatrix}$$

Proof. Lets examine the sine wave $f(x)=a\sin(bx-c)$ as a 7th order Bezier path. First we need to write $f(x)=a\sin(bx-c)$ in Maclaurin series expansion. For sine function 7th degree Maclaurin series expansion is:

$$f(x)=\sum_{n=0}^{7}[a\sin(bx-c)]^{(n)}(0)\frac{x^n}{n!}=$$

$$=-a(\sin c)+ab(\cos c)x+\frac{ab^2\sin c}{2!}x^2+\frac{-ab^3(\cos c)}{3!}x^3+$$

$$+\frac{-ab^4(\sin c)}{4!}x^4+\frac{ab^5(\cos c)}{5!}x^5+\frac{ab^6(\sin c)}{6!}x^6+\frac{-ab^7(\cos c)}{7!}x^7$$

It can be written as in parametric form and a 7th degree polynomial function:

$$[t,a\sin(bt-c)]=\left(t,\frac{-ab^7\cos c}{7!}t^7+\frac{ab^6\sin c}{6!}t^6+\frac{ab^5\cos c}{5!}t^5+\right.$$

$$\left.+\frac{-ab^4\sin c}{4!}t^4+\frac{-ab^3\cos c}{3!}t^3+\frac{ab^2\sin c}{2!}t^2+abt(\cos c)-a\sin c\right)$$

Also this can be written in matrix form with the matrix representation of 7th order Bezier path as in:

$$[t,a\sin(bt-c)]=(t,a_7t^7+a_6t^6+a_5t^5+a_4t^4+a_3t^3+a_2t^2+a_1t+a_0)$$

Hence we have:

$$[t,a\sin(bt-c)]=\begin{bmatrix}t^7\\t^6\\t^5\\t^4\\t^3\\t^2\\t\\1\end{bmatrix}\begin{bmatrix}0&0&0&0&0&0&1&0\\-\frac{\text{ab}^7\cos c}{7!}&\frac{ab^6\sin c}{6!}&\frac{ab^5\cos c}{5!}&\frac{-ab^4\sin c}{4!}&\frac{-ab^3\cos c}{3!}&\frac{ab^2\sin c}{2!}&\text{abcosc}&-\text{asinc}\end{bmatrix}$$

$$=[t^7\quad t^6\quad t^5\quad t^4\quad t^3\quad t^2\quad t\quad 1][B^7][P_0\quad P_1\quad P_2\quad P_3\quad P_4\quad P_5\quad P_6\quad P_7]^T$$

Solving the equation we get the control points as in the result of the matrix product:

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix} = [B^7]^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\dfrac{\text{ab}^7 cosc}{7!} & \dfrac{ab^6 \sin c}{6!} & \dfrac{ab^5 \cos c}{5!} & \dfrac{-ab^4 \sin c}{4!} & \dfrac{-ab^3 \cos c}{3!} & \dfrac{ab^2 \sin c}{2!} & \text{abcosc} & -\text{asinc} \end{bmatrix}^T$$

Corollary 4. The apscissas and ordinates of the control points of sine wave $\alpha(t) = [t, a\sin(bt-c)]$ as a 7th order Bezier path are given in the following way respectivly, where $[B^7]^{-1}$ is inverse of any 7th order Bezier path matrix:

$$x_0 = 0 \; x_1 = \frac{1}{7} \; x_2 = \frac{2}{7} \; x_3 = \frac{3}{7} \; x_4 = \frac{4}{7} \; x_5 = \frac{5}{7} \; x_6 = \frac{6}{7} \; x_7 = 1$$

and

$$[y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 \quad y_6 \quad y_7]^T =$$
$$= [B^7]^{-1} \left[-\frac{ab^7 \cos c}{7!} \frac{ab^6 \sin c}{6!} \frac{ab^5 \cos c}{5!} \frac{-ab^4 \sin c}{4!} \frac{-ab^3 \text{cos} c}{3!} \frac{ab^2 \text{sin} c}{2!} ab\text{cos}c - a\text{sin}c \right]$$

Corollary 5. The coefficients $[t, a\sin(bt-c)]$ of based on the ordinates of control points of the 7th order Bezier path are:

$$\begin{bmatrix} -\dfrac{ab^7 \cos c}{7!} \\ \dfrac{ab^6 \sin c}{6!} \\ \dfrac{ab^5 \cos c}{5!} \\ \dfrac{-ab^4 \sin c}{4!} \\ \dfrac{4!}{4!}\cos c \\ \dfrac{-ab^2 \sin c}{2!} \\ ab \cos c \\ -a \sin c \end{bmatrix} = \begin{bmatrix} 7y_1 - y_0 - 21y_2 + 35y_3 - 35y_4 + 21y_5 - 7y_6 + y_7 \\ 7y_0 - 42y_1 + 105y_2 - 140y_3 + 105y_4 - 42y_5 + 7y_6 \\ 105y_1 - 21y_0 - 210y_2 + 210y_3 - 105y_4 + 21y_5 \\ 35y_0 - 140y_1 + 210y_2 - 140y_3 + 35y_4 \\ 105y_1 - 35y_0 - 105y_2 + 35y_3 \\ 21y_0 - 42y_1 + 21y_2 \\ 7y_1 - 7y_0 \\ y_0 \end{bmatrix}$$

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