HOW TO APPROXIMATE COSINE CURVE WITH 4TH AND 6TH ORDER BEZIER CURVE IN PLANE?

by

Seyda KILICOGLU[a,*] and Semra YURTTANCIKMAZ[b]

[a] Department of Mathematics, Faculty of Education, Baskent University, Ankara, Turkey
[b] Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, Turkey

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There are many ways to approximate cosine curve. In this study we have examined the way how the cosine curve can be written as any order Bezier curve. As a result using the Maclaurin series we have examined cosine curve as the 4th and the 6th order Bezier curve based on the control points with matrix form in $\mathbf{E}^2$*. We give the control points of the 4th and the 6th order Bezier curve based on the coefficients. Also we give the coefficients based on the the control points of the 4th and the 6th order Bezier curve too.*

Key words: *cosine curve, 4th order Bezier curve, 6th order Bezier curve, Maclaurin series*

Introduction and preliminaries

A Bezier curve is frequently used in computer graphics and related fields, in vector graphics, used in animation as a tool to control motion [1, 2]. In animation applications such as Adobe Flash and Synfig, Bezier curves are used to outline for example movement. Users outline the wanted path in Bezier curves, and the application creates the needed frames for the object to move along the path. For 3-D animation Bezier curves are often used to define 3-D paths as well as 2-D curves for keyframe interpolation. In [3] A dual unit spherical Bézier-like curve corresponds to a ruled surface by using Study's transference principle and closed ruled surfaces are determined via control points and also, integral invariants of these surfaces are investigated. Researchers have written many publications on Bezier curves, but some of these studies inspired this article. For example: In [4], Bezier curves with curvature and torsion continuity has been examined. In [5, 6], Bezier curves and surfaces has been given. In [7], Bezier curves are designed for Computer-Aided Geometric. Recently equivalence conditions of control points and application to planar Bezier curves have been examined. In [8], Frenet apparatus of the cubic Bezier curves has been examined in $\mathbf{E}^3$. In [9], A cubic trigonometric Bezier-like curve similar to the cubic B´ezier curve, with a shape parameter, is presented. In here, first 5th order Bezier curve and its first, second and third derivatives have been examined based on the control points of 5th order Bezier curve in $\mathbf{E}^3$. We have already examine in cubic Bezier curves and involutes in [8, 10]. The Bertrand and the Mannheim mate of a cubic Bezier curve by using matrix representation have been researhed in $\mathbf{E}^3$ [11, 12], respectively. In [13], it has been examined the 5th order Bezier curve and its derivatives. In [14], it has been

* Corresponding author, e-mail: seyda@baskent.edu.tr

researched the answer of the question *How to find a n^{th} order Bezier curve if we know the first, second and third derivatives*?

Generally n^{th} order Bezier curves can be defined by $n+1$ control points $P_0, P_1, \ldots, P_n$ with the parametrization:

$$\mathbf{B}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} [P_i]$$

We have already known that the matrix representation of any 4th order Bezier curve $\alpha(t) = (t, a_4 t^4 + a_3 t^3 + a_2 t^2 + a t_1 + a_0)$ in $\mathbf{E}^2$ is:

$$\alpha(t) = [t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^4][P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4]^T$$

where the coefficient matrix and the inverse matrix of 4th order Bezier curves matrix are:

$$[B^4] = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } [B^4]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{6} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

with the control points:

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4]^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}^T$$

For more detail see [8, 15].

It is well known that Taylor series of a function is an infinite sum of the functions derivatives at a single point *a*, also a Maclaurin series is a taylor series where $a = 0$. For any function Taylor series expansion is:

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}$$

also a Maclaurin series is a taylor series where $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$$

In this study we will focus on the 4th and 6th order Bezier curves in $\mathbf{E}^2$.

Cosine curve as a 4th order Bezier curve

First let us examine the cosine curve of the function $f(x) = \cos x$ as a 4th order Bezier curve.

Theorem 1. The matrix representation of cosine curve $f(x)=\cos x$ as a 4th order Bezier curve is:

$$(t,\cos t)=[t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^4]\begin{bmatrix} 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \\ 1 & 1 & \frac{11}{12} & \frac{3}{4} & \frac{13}{24} \end{bmatrix}^T$$

with the control points:

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4]^T=\begin{bmatrix} 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \\ 1 & 1 & \frac{11}{12} & \frac{3}{4} & \frac{13}{24} \end{bmatrix}^T$$

Proof. For cosine function, the 4th degree Maclaurin series expansion is $\cos x = 1 - x^2/2! + x^4/4!,$ it can be written as in the parametric form and a 5th degree polynomial function:

$$(t,\cos t)=\left(t,\frac{t^4}{4!}-\frac{t^2}{2!}+1\right)=(t,a_4t^4+a_3t^3+a_2t^2+a_1t+a_0)$$

Also this can be written in matrix form with the matrix representation of 4th order Bezier curve as in:

$$(t,\cos t)=\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{vmatrix} 0 & \frac{1}{4!} \\ 0 & 0 \\ 0 & \frac{-1}{2!} \\ 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^4] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

Solving the equation we get the control points $P_0, P_1, P_2, P_3,$ and P_4.

Corollary 1. The apscissas and the ordinates of the control points of cosine curve as a 4th order Bezier curve are:

$$x_0=0 \; x_1=\frac{1}{4} \; x_2=\frac{2}{4} \; x_3=\frac{3}{4} \; x_4=1$$

and

$$[y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4]^T=[B^4]^{-1}\begin{bmatrix} \frac{1}{4!} & 0 & \frac{-1}{2!} & 0 & 1 \end{bmatrix}^T$$

Now, let's examine the cosine curve as a 4th order Bezier curve. First we will examine the cosine curve $f(x)=a\cos bx$.

Theorem 2. The matrix representation of the cosine curve of the function $f(x)=a\cos bx$ as a 4th order Bezier curve is:

$$(t, a\cos bt)=[t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^4]\begin{bmatrix} 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \\ a & a & a-\frac{ab^2}{12} & a-\frac{ab^2}{4} & \frac{ab^4}{24}-\frac{ab^2}{2}+a \end{bmatrix}^T$$

where control points $P_0, P_1, P_2, P_3,$ and P_4 are:

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4]^T=\begin{bmatrix} 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \\ a & a & a-\frac{ab^2}{12} & a-\frac{ab^2}{4} & \frac{ab^4}{24}-\frac{ab^2}{2}+a \end{bmatrix}^T$$

Proof. We need to write $f(x)=a\cos bx$ in Maclaurin series expansion. For cosine function $f(x)=a\cos bx,$ as any 4th degree Maclaurin series expansion is:

$$\begin{aligned} f(x) &= \sum_{n=0}^{4}(a\cos bx)^{(n)}(0)\frac{x^n}{n!} \\ &= a-\frac{ab^2}{2!}x^2+\frac{ab^4\cos(0)}{4!}x^4 \end{aligned}$$

This 4th degree polynomial function can be written as in parametric form

$$(t, a\cos bt)=\left(t, \frac{ab^4}{4!}t^4-\frac{ab^2}{2!}t^2+a\right)=(t, a_4t^4+a_3t^3+a_2t^2+a_1t+a_0)$$

Also this can be written in matrix form with the matrix representation of 4th order Bezier curve as in:

$$(t, a\cos bt)=\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{vmatrix} 0 & \frac{ab^4}{4!} \\ 0 & 0 \\ 0 & -\frac{ab^2}{2!} \\ 1 & 0 \\ 0 & a \end{vmatrix} = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^4]\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

Hence solving the equation as in the following way:

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4]^T=[B^4]^{-1}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ \frac{ab^4}{4!} & 0 & \frac{-ab^2}{2!} & 0 & a \end{bmatrix}^T$$

So, we get the proof.

Now also, we will examine sine function $f(x) = a\cos(bx - c)$ as a 4th order Bezier curve.

Theorem 3. The matrix representation of the cosine curve of $f(x) = a\cos(bx - c)$ as a 4th order Bezier curve is:

$$[t, a\cos(bt-c)] = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^4] \begin{bmatrix} 0 & a\cos c \\ \frac{1}{4} & a\cos c + \frac{ab}{4}\sin c \\ \frac{1}{2} & -\frac{ab^2}{12}\cos c + \frac{ab}{2}\sin c + a\cos c \\ \frac{3}{4} & -\frac{ab^3}{24}\sin c - \frac{ab^2}{4}\cos c + \frac{3ab}{4}\sin c + a\cos c \\ 1 & \frac{ab^4\cos c}{24} - \frac{ab^3}{6}\sin c - \frac{ab^2}{2}\cos c + ab\sin c + a\cos c \end{bmatrix}$$

where the control points $P_0, P_1, P_2, P_3,$ and P_4 are

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} 0 & a\cos c \\ \frac{1}{4} & a\cos c + \frac{ab}{4}\sin c \\ \frac{1}{2} & -\frac{ab^2}{12}\cos c + \frac{ab}{2}\sin c + a\cos c \\ \frac{3}{4} & -\frac{ab^3}{24}\sin c - \frac{ab^2}{4}\cos c + \frac{3ab}{4}\sin c + a\cos c \\ 1 & \frac{ab^4\cos c}{24} - \frac{ab^3}{6}\sin c - \frac{ab^2}{2}\cos c + ab\sin c + a\cos c \end{bmatrix}$$

Proof. Lets examine the cosine curve as a 4th order Bezier curve. First we need to write $f(x) = a\cos(bx - c)$ in Maclaurin series expansion. For cosine function 4th degree Maclaurin series expansion is:

$$f(x) = \sum_{n=0}^{4} [a\cos(bx-c)]^{(n)}(0)\frac{x^n}{n!} =$$

$$= [a\cos(b.0-c)]1 + [a\cos(bx-c)]^{'}(0)x + [a\cos(bx-c)]^{''}(0)\frac{x^2}{2!} +$$

$$+[a\cos(bx-c)]^{'''}(0)\frac{x^3}{3!} + [a\cos(bx-c)]^{(2v)}(0)\frac{x^4}{4!}$$

This 4th degree polynomial function can be written as in parametric form

$$[t, a\cos(bt-c)] = \left[t, \frac{ab^4\cos c}{4!}t^4 - \frac{ab^3\sin c}{3!}t^3 - \frac{ab^2\cos c}{2!}t^2 + (ab\sin c)t + a\cos c \right]$$

$$= (t, a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0)$$

Also this can be written in matrix form with the matrix representation of 4th order Bezier curve as in:

$$[t, a\cos(bt-c)] = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{vmatrix} 0 & \dfrac{ab^4\cos c}{4!} \\ 0 & \dfrac{-ab^3\sin c}{3!} \\ 0 & \dfrac{-ab^2\cos c}{2!} \\ 1 & ab\sin c \\ 0 & a\cos c \end{vmatrix} = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^4] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

Solving the equation we get the control points $P_0, P_1, P_2, P_3,$ and P_4 as in the result of the matrix product:

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4]^T = [B^4]^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ \dfrac{ab^4\cos c}{4!} & -\dfrac{ab^3\sin c}{3!} & \dfrac{-ab^2\cos c}{2!} & ab\sin c & a\cos c \end{bmatrix}^T$$

This completes the proof.

Corollary 1. The coefficients of the $[t, a\cos(bt-c)]$ based on the control points of the 4th order Bezier curve as:

$$\begin{vmatrix} 0 & \dfrac{ab^4\cos c}{4!} \\ 0 & \dfrac{-ab^3\sin c}{3!} \\ 0 & \dfrac{-ab^2\cos c}{2!} \\ 1 & ab\sin c \\ 0 & a\cos c \end{vmatrix} = \begin{bmatrix} x_0 - 4x_1 + 6x_2 - 4x_3 + x_4 & y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 \\ 12x_1 - 4x_0 - 12x_2 + 4x_3 & 12y_1 - 4y_0 - 12y_2 + 4y_3 \\ 6x_0 - 12x_1 + 6x_2 & 6y_0 - 12y_1 + 6y_2 \\ 4x_1 - 4x_0 & 4y_1 - 4y_0 \\ x_0 & y_0 \end{bmatrix}$$

Cosine curve as a 6th order Bezier curve

We have to write the coefficients matrix of any 6th order Bezier curve. We have already known that the matrix representation is 6th order Bezier curve as follows.

Theorem 4. The coefficients matrix and inverse matrix of any 6th order Bezier curve are:

$$[B^6] = \begin{vmatrix} 1 & -6 & 15 & -20 & 15 & -6 & 1 \\ -6 & 30 & -60 & 60 & -30 & 6 & 0 \\ 15 & -60 & 90 & -60 & 15 & 0 & 0 \\ -20 & 60 & -60 & 20 & 0 & 0 & 0 \\ 15 & -30 & 15 & 0 & 0 & 0 & 0 \\ -6 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

and

$$[B^6]^{-1}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{15} & \frac{1}{3} & 1 \\ 0 & 0 & 0 & \frac{1}{20} & \frac{1}{5} & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{15} & \frac{1}{5} & \frac{2}{5} & \frac{2}{3} & 1 \\ 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{5}{6} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Theorem 5. The matrix representation of cosine curve $f(x)=\cos x$ as a 6th order Bezier curve based on the coefficients is:

$$(t,\cos t)=[t^6 \quad t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^6]\begin{bmatrix} 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{5}{6} & 1 \\ 1 & 1 & \frac{29}{30} & \frac{9}{10} & \frac{289}{360} & \frac{49}{72} & \frac{389}{720} \end{bmatrix}^T$$

Proof. For cosine function 6th degree Maclaurin series expansion is:

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}$$

It can be written as in parametric form:

$$(t,a\cos bt)=\left(t,-\frac{1}{6!}t^6+\frac{1}{4!}t^4-\frac{1}{2!}t^2+1\right)=(t,a_6t^6+a_5t^5+a_4t^4+a_3t^3+a_2t^2+a_1t+a_0)$$

Also this can be written in matrix form with the matrix representation of 6th order Bezier curve as in:

$$(t,a\cos bt)=\begin{bmatrix} t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{vmatrix} 0 & -\frac{1^6}{6!} \\ 0 & 0 \\ 0 & \frac{1^4}{4!} \\ 0 & 0 \\ 0 & -\frac{1^2}{2!} \\ 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{bmatrix} t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^6] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix}$$

Hence solving the equation we get the control points as in the following way:

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5 \quad P_6]^T = [B^6]^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{1^6}{6!} & 0 & \frac{1^4}{4!} & 0 & -\frac{1^2}{2!} & 0 & 1 \end{bmatrix}^T$$

Corollary 2. The apscissas and ordinates of the control points of cosine curve as a 6th order Bezier curve are:

$$x_0 = 0, \quad x_1 = \frac{1}{6}, \quad x_2 = \frac{2}{6}, \quad x_3 = \frac{3}{6}, \quad x_4 = \frac{4}{6}, \quad x_5 = \frac{5}{6}, \quad x_6 = 1$$

$$[y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 \quad y_6]^T = [B^6]^{-1} \begin{bmatrix} -\frac{1^6}{6!} & 0 & \frac{1^4}{4!} & 0 & -\frac{1^2}{2!} & 0 & 1 \end{bmatrix}^T$$

Theorem 6. The matrix representation of the cosine curve of function $f(x) = a\cos bx$ as a 6th order Bezier curve is:

$$(t, a\cos bt) = \begin{bmatrix} t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^6] \begin{bmatrix} 0 & a \\ \frac{1}{6} & a - \frac{1}{30}ab^2 \\ \frac{1}{3} & a - \frac{1}{10}ab^2 \\ \frac{1}{2} & \frac{1}{360}ab^4 - \frac{1}{5}ab^2 + a \\ \frac{2}{3} & \frac{1}{72}ab^4 - \frac{1}{3}ab^2 + a \\ \frac{5}{6} & -\frac{1}{720}ab^6 + \frac{1}{24}ab^4 - \frac{1}{2}ab^2 + a \\ 1 & -\cos \end{bmatrix}$$

with the control points:

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = \begin{bmatrix} 0 & a \\ \frac{1}{6} & a \\ \frac{1}{3} & a - \frac{1}{30}ab^2 \\ \frac{1}{2} & a - \frac{1}{10}ab^2 \\ \frac{2}{3} & \frac{1}{360}ab^4 - \frac{1}{5}ab^2 + a \\ \frac{5}{6} & \frac{1}{72}ab^4 - \frac{1}{3}ab^2 + a \\ 1 & -\frac{1}{720}ab^6 + \frac{1}{24}ab^4 - \frac{1}{2}ab^2 + a \end{bmatrix}.$$

Proof. For cosine function $f(x) = a\cos bx,$ can be written as 6th degree Maclaurin series expansion as:

$$f(x) = \sum_{n=0}^{6} (a\cos bx)^{(n)}(0)\frac{x^n}{n!} =$$

$$= a - \frac{ab^2}{2!}x^2 + \frac{ab^4\cos(0)}{4!}x^4 + \frac{-ab^5\sin(0)}{5!}x^5 + \frac{-ab^6\cos(0)}{6!}x^6$$

$$f(x) = a - \frac{ab^2x^2}{2!} + \frac{ab^4x^4}{4!} - \frac{ab^6x^6}{6!}$$

This 6th degree polynomial function can be written as in parametric form:

$$(t, a\cos bt) = \left(t, -\frac{ab^6}{6!}t^6 + \frac{ab^4}{4!}t^4 - \frac{ab^2t^2}{2!} + a\right) =$$

$$= (t, a_6t^6 + a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0)$$

Also this can be written in matrix form with the matrix representation of 6th order Bezier curve as in:

$$(t, a\cos bt) = \begin{bmatrix} t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{ab^6}{6!} \\ 0 & 0 \\ 0 & \frac{ab^4}{4!} \\ 0 & 0 \\ 0 & -\frac{ab^2}{2!} \\ 1 & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix} [B^6] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix}$$

Hence solving the equation, we get the control points:

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = \begin{bmatrix} 0 & a \\ \frac{1}{6} & a - \frac{1}{30}ab^2 \\ \frac{1}{3} & a - \frac{1}{10}ab^2 \\ \frac{1}{2} & \frac{1}{360}ab^4 - \frac{1}{5}ab^2 + a \\ \frac{2}{3} & \frac{1}{72}ab^4 - \frac{1}{3}ab^2 + a \\ \frac{5}{6} & -\frac{1}{720}ab^6 + \frac{1}{24}ab^4 - \frac{1}{2}ab^2 + a \end{bmatrix}$$

This completes the proof.

Theorem 7. The matrix representation of the sine curve of $f(x)=a\cos(bx-c)$ as a 6th order Bezier curve is:

$$[t, a\cos(bt-c)]=[t^6 \quad t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^6][P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5 \quad P_6]^T$$

with the control points:

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix}=\begin{bmatrix} 0 & a\cos c \\ \frac{1}{6} & -\frac{ab^2\cos c}{30}+\frac{ab\sin c}{3}+a\cos c \\ \frac{1}{3} & \frac{-ab^3\sin c}{120}-\frac{ab^2\cos c}{10}+\frac{ab\sin c}{2}+a\cos c \\ \frac{1}{2} & \frac{ab^4\cos c}{360}-\frac{ab^3\sin c}{30}-\frac{ab^2\cos c}{5}+\frac{2ab\sin c}{3}+a\cos c \\ \frac{2}{3} & \frac{-ab^5\sin c}{720}+\frac{ab^4\cos c}{72}-\frac{ab^3\sin c}{12}-\frac{ab^2\cos c}{3}+\frac{5ab\sin c}{6}+a\cos c \\ \frac{5}{6} & \frac{-ab^6\cos c}{720}-\frac{ab^5\sin c}{120}+\frac{ab^4\cos c}{24}-\frac{ab^3\sin c}{6}-\frac{ab^2\cos c}{2}+ab\sin c+a\cos c \end{bmatrix}$$

Proof. Lets examine the cosine curve as a 6th order Bezier curve. First we need to write $f(x)=a\cos(bx-c)$ in Maclaurin series expansion. For cosine function 6th degree Maclaurin series expansion is:

$$f(x)=\sum_{n=0}^{6}[a\cos(bx-c)]^{(n)}(0)\frac{x^n}{n!}=$$
$$=a\cos c+(ab\sin c)x-ab^2(\cos c)\frac{x^2}{2!}-ab^3(\sin c)\frac{x^3}{3!}+$$
$$+ab^4(\cos c)\frac{x^4}{4!}-ab^5(\sin c)\frac{x^5}{5!}-ab^6(\cos c)\frac{x^6}{6!}$$

This 6th degree polynomial function can be written as in parametric form:

$$[t, a\cos(bt-c)] \quad =\left(\frac{ab^7\sin(c)}{7!}t^7, \frac{-ab^6\cos(c)}{6!}t^6-\frac{ab^5\sin(c)}{5!}t^5+\frac{ab^4\cos c}{4!}t^4-\right.$$
$$\left.-\frac{ab^3\sin c}{3!}t^3-\frac{ab^2\cos c}{2!}t^2+(ab\sin c)t+a\cos c\right)$$

Also this can be written in matrix form with the matrix representation of 6th order Bezier curve as in:

$$[t, a\cos(bt-c)] = \begin{bmatrix} t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{vmatrix} 0 & \dfrac{-ab^6\cos c}{6!} \\ 0 & \dfrac{-ab^5\sin c}{5!} \\ 0 & \dfrac{ab^4\cos c}{4!} \\ 0 & \dfrac{-ab^3\sin c}{3!} \\ 0 & \dfrac{-ab^2\cos c}{2!} \\ 1 & ab\sin c \\ 0 & a\cos c \end{vmatrix} = \begin{bmatrix} t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^6] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix}$$

Hence solving the equation we get the proof.

Corollary 3. The coefficients of the $[t, a\cos(bt-c)]$ based on the control points of the 6th order Bezier curve are:

$$\begin{vmatrix} 0 & \dfrac{-ab^6\cos c}{6!} \\ 0 & \dfrac{-ab^5\sin c}{5!} \\ 0 & \dfrac{ab^4\cos c}{4!} \\ 0 & \dfrac{-ab^3\sin c}{3!} \\ 0 & \dfrac{-ab^2\cos c}{2!} \\ 1 & ab\sin c \\ 0 & a\cos c \end{vmatrix} = \begin{bmatrix} x_0 - 6x_1 + 15x_2 - 20x_3 + 15x_4 - 6x_5 + x_6 & y_0 - 6y_1 + 15y_2 - 20y_3 + 15y_4 - 6y_5 + y_6 \\ 30x_1 - 6x_0 - 60x_2 + 60x_3 - 30x_4 + 6x_5 & 30y_1 - 6y_0 - 60y_2 + 60y_3 - 30y_4 + 6y_5 \\ 15x_0 - 60x_1 + 90x_2 - 60x_3 + 15x_4 & 15y_0 - 60y_1 + 90y_2 - 60y_3 + 15y_4 \\ 60x_1 - 20x_0 - 60x_2 + 20x_3 & 60y_1 - 20y_0 - 60y_2 + 20y_3 \\ 15x_0 - 30x_1 + 15x_2 & 15y_0 - 30y_1 + 15y_2 \\ 6x_1 - 6x_0 & 6y_1 - 6y_0 \\ x_0 & y_0 \end{bmatrix}$$

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