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AN APPLICATION ON DIFFERENTIAL EQUATIONS OF ORDER m

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Abstract. In this paper we introduce the classes $\mathcal{T}_n(p,\lambda,A,B)$ and $\mathcal{K}_n(p,\lambda,\mu,m,A,B)$ and derive distortion inequalities of the functions belonging to class $\mathcal{K}_n(p,\lambda,\mu,m,A,B)$. Further we apply to the $(n,\delta)-$ neighborhoods of functios in the class $\mathcal{K}_n(p,\lambda,\mu,m,A,B)$.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{T}_n(p)$ denote the class of functions $f(z)$ normalized by

$$f(z)=z^p-\sum_{k=n+p}^{\infty}a_kz^k \qquad (a_k\geq 0;\ n,p\in\mathbb{N}:=\{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ on the complex plane $\mathbb{C}$.

Let f and F be analytic functions in the unit disk $\mathbb{U}$. A function f is said to be subordinate to F, written as $f\prec F$ or $f(z)\prec F(z)$, if there exists a Schwarz function $\omega:\mathbb{U}\to\mathbb{U}$ with $\omega(0)=0$ such that $f(z)=F\big(\omega(z)\big)$. In particular, if F is univalent in $\mathbb{U}$, we have the following equivalence:

$$f(z)\prec F(z)\Longleftrightarrow [f(0)=F(0)\wedge f(\mathbb{U})\subseteq F(\mathbb{U})].$$

Following the earlier investigations by Goodman [11] and Ruscheweyh [15] (see also [1–3, 5, 6, 9, 13]), we define the $(n,\delta)-$neighborhoods of functions $f\in\mathcal{T}_n(p)$ by

$$\mathcal{N}_{n,\delta}(f;g)=\left\{g\in\mathcal{T}_n(p):g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k \text{ and } \sum_{k=n+p}^{\infty}k\,|a_k-b_k|\leq\delta\right\}. \tag{1.2}$$

Let $\mathcal{S}^*$ and $\mathcal{C}$ be the usual subclasses of functions which members are univalent, starlike and convex in $\mathbb{U}$, respectively.

A function $f \in \mathcal{T}_n(p)$ is called p-valently starlike of order γ if it satisfies the conditions

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma \tag{1.3}$$

and

$$\int_0^{2\pi} \Re\left(\frac{zf'(z)}{f(z)}\right) d\theta = 2p\pi \tag{1.4}$$

for $0 \leq \gamma < p,\ p \in \mathbb{N}$ and $z \in \mathbb{U}$. We denote by $\mathcal{S}_n^*(p,\gamma)$ the class of all p-valently starlike functions of order γ. Furthermore, a function $f \in \mathcal{T}_n(p)$ is called p-valently convex of order γ if it satisfies the conditions

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma \tag{1.5}$$

and

$$\int_0^{2\pi} \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) d\theta = 2p\pi \tag{1.6}$$

for $0 \leq \gamma < p,\ p \in \mathbb{N}$ and $z \in \mathbb{U}$. We denote by $\mathcal{C}_n(p,\gamma)$ the class of all p-valently convex functions of order γ.

Clearly, $\mathcal{S}^* := \mathcal{S}_1^*(1,0)$ and $\mathcal{C} := \mathcal{C}_1(1,0)$. We note that

$$f(z) \in \mathcal{C}_n(p,\gamma) \Leftrightarrow \frac{f'(z)}{p} \in \mathcal{S}_n^*(p,\gamma) \tag{1.7}$$

The classes $\mathcal{S}_n^*(p,\gamma)$ and $\mathcal{C}_n(p,\gamma)$ were introduced by Patil and Thakare [14].

Therefore, various subclasses of p-valent functions in $\mathbb{U}$ was studied by Altıntaş et al. in [8], Nunokawa et al. in [12] and Srivastava et al. in [16, 17].

A function $f \in \mathcal{T}_n(p)$ is called Janowski p-valently starlike if it satisfies the condition

$$\frac{zf'(z)}{f(z)} \prec p\frac{1+Az}{1+Bz} \tag{1.8}$$

for $-1 \leq A < B \leq 1,\ p \in \mathbb{N}$ and $z \in \mathbb{U}$. We denote by $\mathcal{S}_n^*(p,A,B)$ the class of all Janowski p-valently starlike functions.

Also, a function $f \in \mathcal{T}_n(p)$ is called Janowski p-valently convex if it satisfies the condition

$$1 + \frac{zf''(z)}{f'(z)} \prec p\frac{1+Az}{1+Bz} \tag{1.9}$$

for $-1 \leq A < B \leq 1,\ p \in \mathbb{N}$ and $z \in \mathbb{U}$. We denote by $\mathcal{C}_n(p,A,B)$ the class of all Janowski p-valently convex functions.

We note that, $\mathcal{S}_n^*(p,\gamma) := \mathcal{S}_n^*(p,1-2\gamma,-1)$, $\mathcal{S}^* := \mathcal{S}_1^*(1,1,-1)$ and $\mathcal{C}_n(p,\gamma) := \mathcal{C}_n(p,1-2\gamma,-1)$, $\mathcal{C} := \mathcal{C}_1(1,1,-1)$.

Let $\mathcal{T}_n(p,\lambda,A,B)$ denote the subclass of $\mathcal{T}_n(p)$ consisting of functions $f(z)$ which satisfy the following inequality:

$$\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)} \prec p\frac{1+Az}{1+Bz} \tag{1.10}$$

where $0\leq\lambda\leq 1$, $-1\leq A<B\leq 1$, $p\in\mathbb{N}$, $z\in\mathbb{U}$. The class $\mathcal{T}_n(p,\lambda,A,B)$ was introduced and studied by Altıntaş in [3, 7].

Clearly, we have the following relationships:

$$\mathcal{S}_n^*(p,A,B) := \mathcal{T}_n(p,0,A,B) \text{ and } \mathcal{C}_n(p,A,B) := \mathcal{T}_n(p,1,A,B).$$

We note that these classes are studied in [10].

Recently, we have defined and studied in [1, 2, 4–6] the following second order differential equation:

$$z^2\frac{d^2w}{dz^2}+2(\mu+1)z\frac{dw}{dz}+\mu(\mu+1)w=(p+\mu)(p+\mu+1)g \tag{1.11}$$

where $w=f(z)\in\mathcal{T}_n(p)$, $g=g(z)$ satisfy the following inequality:

$$\Re\frac{zg'(z)+\lambda z^2g''(z)}{\lambda zg'(z)+(1-\lambda)g(z)}>\alpha \tag{1.12}$$

where $0\leq\lambda\leq 1, 0\leq\alpha<1, p\in\mathbb{N}, \mu>-p, z\in\mathbb{U}$.

Definition 1. The following non-homogenous Cauchy-Euler differential equation of order 3 is

$$\begin{aligned} z^3\frac{d^3w}{dz^3}+3(\mu+2)z^2\frac{d^2w}{dz^2}+3(\mu+1)(\mu+2)z\frac{dw}{dz}+\mu(\mu+1)(\mu+2)w \\ =(p+\mu)(p+\mu+1)(p+\mu+2)g \end{aligned} \tag{1.13}$$

where $w=f(z)\in\mathcal{T}_n(p)$, $g=g(z)\in\mathcal{T}_n(p,\lambda,A,B)$ and $\mu>-p$.

This differential equation is defined and studied in [3].

Definition 2. The following non-homogenous Cauchy-Euler differential equation of order m is

$$\begin{aligned} z^m\frac{d^mw}{dz^m}+\binom{m}{1}(\mu+m-1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}}+\ldots+\binom{m}{r}\prod_{j=r}^{m-1}(\mu+j)z^r\frac{d^rw}{dz^r}+ \\ \ldots+\binom{m}{m}\prod_{j=0}^{m-1}(\mu+j)w=\prod_{j=0}^{m-1}(p+\mu+j)g \end{aligned} \tag{1.14}$$

where $w=f(z)\in\mathcal{T}_n(p)$, $g=g(z)\in\mathcal{T}_n(p,\lambda,A,B)$, $m\in\mathbb{N}^*:=\{2,3,\cdots\}$ and $\mu>-p$.

Finally $\mathcal{K}_n(p,\lambda,\mu,m,A,B)$ denote the subclass of the class $\mathcal{T}_n(p)$ consisting of functions $f(z)$, satisfying the equation (1.14) in Definition 2.

In this paper, we obtain coefficient bounds, distortion inequalities and $(n,\delta)-$ neighborhoods of functions $f\in\mathcal{T}_n(p)$ in the class $\mathcal{K}_n(p,\lambda,\mu,m,A,B)$.

2. Coefficient Bounds and Distortion Inequalities

For proving the main results in this paper, we will use the following lemmas.

Lemma 1 ([3]). *Let the function $\mathcal{T}_n(p)$ be defined by (1.1). Then $f(z)$ is in the class $\mathcal{T}_n(p,\lambda,A,B)$ if and only if*

$$\sum_{k=n+p}^{\infty}(k-p-pA+kB)(\lambda k-\lambda+1)a_k \leq p(B-A)(\lambda p-\lambda+1) \tag{2.1}$$

where $0\leq\lambda\leq 1\,,-1\leq A<B\leq 1,\ p\in\mathbb{N}$.

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{p(B-A)(\lambda p-\lambda+1)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]}z^{n+p}. \tag{2.2}$$

Lemma 2 ([3]). *Let the function $f(z)\in\mathcal{T}_n(p)$ defined by (1.1) be in the class $\mathcal{T}_n(p,\lambda,A,B)$. Then, we have*

$$\sum_{k=n+p}^{\infty}a_k\leq\frac{p(B-A)(\lambda p-\lambda+1)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]} \tag{2.3}$$

and

$$\sum_{k=n+p}^{\infty}ka_k\leq\frac{p(B-A)(\lambda p-\lambda+1)(n+p)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]}. \tag{2.4}$$

The distortion inequalities for functions in the class $\mathcal{K}_n(p,\lambda,\mu,m,A,B)$ are given by Theorem 1 below.

Theorem 1. *If a function $f\in\mathcal{T}_n(p)$ is in the class $\mathcal{K}_n(p,\lambda,\mu,m,A,B)$, then*

$$|f(z)|\leq|z|^p+\frac{p(B-A)(\lambda p-\lambda+1)\prod_{j=0}^{m-1}(p+\mu+j)}{(m-1)[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]\prod_{j=0}^{m-2}(n+p+\mu+j)}|z|^{n+p} \tag{2.5}$$

and

$$|f(z)|\geq|z|^p-\frac{p(B-A)(\lambda p-\lambda+1)\prod_{j=0}^{m-1}(p+\mu+j)}{(m-1)[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]\prod_{j=0}^{m-2}(n+p+\mu+j)}|z|^{n+p}. \tag{2.6}$$

Proof. We first suppose that a function $f\in\mathcal{T}_n(p)$ is in the class $\mathcal{K}_n(p,\lambda,\mu,m,A,B)$. Let the function $g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k\in\mathcal{T}_n(p,\lambda,A,B)$ occurring in the non-homogenous Cauchy-Euler differential equation of order m in (1.14) with, of course,

$$b_k\geq 0 \qquad (k=n+p,n+p+1,\ldots).$$

Then, we readily find from (1.14) that

$$a_k = \frac{\prod_{j=0}^{m-1}(p+\mu+j)}{\prod_{j=0}^{m-1}(k+\mu+j)} b_k \qquad (k = n+p, n+p+1, \ldots). \tag{2.7}$$

so that

$$f(z) = z^p - \sum_{k=n+p}^{\infty} \frac{\prod_{j=0}^{m-1}(p+\mu+j)}{\prod_{j=0}^{m-1}(k+\mu+j)} b_k z^k. \tag{2.8}$$

Since $g \in \mathcal{T}_n(p,\lambda,A,B)$, the first assertion (2.3) of Lemma 2 yields the following inequality:

$$b_k \leq \frac{p(B-A)(\lambda p - \lambda + 1)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) - \lambda + 1]}. \tag{2.9}$$

Together with (2.8) and (2.9) yields that

$$|f(z)| \leq |z|^p + |z|^{n+p} \frac{p(B-A)(\lambda p - \lambda + 1)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) - \lambda + 1]} \sum_{k=n+p}^{\infty} \frac{\prod_{j=0}^{m-1}(p+\mu+j)}{\prod_{j=0}^{m-1}(k+\mu+j)} \tag{2.10}$$

and using the following identity that

$$\sum_{k=n+p}^{\infty} \frac{1}{\prod_{j=0}^{m-1}(k+\mu+j)} = \frac{1}{(m-1)!} \sum_{k=n+p}^{\infty} \left[\frac{\binom{m-1}{0}}{k+\mu} - \frac{\binom{m-1}{1}}{k+\mu+1} + \cdots + (-1)^{m-1} \frac{\binom{m-1}{m-1}}{k+\mu+m-1} \right] = \frac{1}{m-1} \frac{1}{\prod_{j=0}^{m-2}(n+p+\mu+j)} \tag{2.11}$$

where $\mu \in \mathbb{R} \setminus \{-n-p, -n-p-1, \ldots\}$. The assertion (2.5) of Theorem 1 follows at once from (2.10) with (2.11). The assertion (2.6) of Theorem 1 can be proven by similarly. □

Corollary 1 ([3]). *If $f \in \mathcal{K}_n(p,\lambda,\mu,2,A,B)$, then we have*

$$|f(z)| \leq |z|^p + \frac{p(B-A)(\lambda p - \lambda + 1)(p+\mu)(p+\mu+1)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) - \lambda + 1](n+p+\mu)} |z|^{n+p}$$

and

$$|f(z)| \geq |z|^p - \frac{p(B-A)(\lambda p - \lambda + 1)(p+\mu)(p+\mu+1)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) - \lambda + 1](n+p+\mu)} |z|^{n+p}.$$

Corollary 2. *If $f \in K_n(p,\lambda,\mu,3,A,B)$, then we have*

$$|f(z)| \leq |z|^p + \frac{p(B-A)(\lambda p-\lambda+1)(p+\mu)(p+\mu+1)(p+\mu+2)}{2[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1](n+p+\mu)(n+p+\mu+1)}|z|^{n+p}$$

and

$$|f(z)| \geq |z|^p - \frac{p(B-A)(\lambda p-\lambda+1)(p+\mu)(p+\mu+1)(p+\mu+2)}{2[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1](n+p+\mu)(n+p+\mu+1)}|z|^{n+p}.$$

3. Neighborhoods for The Class $\mathcal{K}_n(p,\lambda,\mu,m,A,B)$

In this section, we determine inclusion relations for the class $\mathcal{K}_n(p,\lambda,\mu,m,A,B)$ concerning the $(n,\delta)-$neighborhoods defined by (1.2).

Theorem 2. *If $f \in \mathcal{T}_n(p)$ is in the class $\mathcal{K}_n(p,\lambda,\mu,m,A,B)$, then*

$$\mathcal{K}_n(p,\lambda,\mu,m,A,B) \subset N_{n,\delta}(g;f) \tag{3.1}$$

where $g(z)$ is given by (1.14) and

$$\delta := \frac{p(B-A)(\lambda p-\lambda+1)(n+p)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]} \left[1+\frac{\prod_{j=0}^{m-1}(p+\mu+j)}{(m-1)\prod_{j=0}^{m-2}(n+p+\mu+j)}\right]. \tag{3.2}$$

Proof. Suppose that $\mathcal{K}_n(p,\lambda,\mu,m,A,B)$. Then, upon substituting from (2.7) into the following coefficient inequality:

$$\sum_{k=n+p}^{\infty} k|b_k-a_k| \leq \sum_{k=n+p}^{\infty} kb_k + \sum_{k=n+p}^{\infty} ka_k \qquad (a_k \geq 0,\ b_k \geq 0) \tag{3.3}$$

we obtain that

$$\sum_{k=n+p}^{\infty} k|b_k-a_k| \leq \sum_{k=n+p}^{\infty} kb_k + \sum_{k=n+p}^{\infty} \frac{\prod_{j=0}^{m-1}(p+\mu+j)}{\prod_{j=0}^{m-1}(k+\mu+j)}kb_k. \tag{3.4}$$

Since $g \in \mathcal{T}_n(p,\lambda,A,B)$, the second assertion (2.4) of Lemma 2 yields that

$$kb_k \leq \frac{p(B-A)(\lambda p-\lambda+1)(n+p)}{[(n+p)(1+B)-p(1+A)][\lambda(n+p)-\lambda+1]} \quad (k=n+p,n+p+1,\ldots). \tag{3.5}$$

In the right hand side of (3.4), we obtain the assertion (3.2) using (3.5) and (2.11), respectively.

Thus, by Definition 2 with $g(z)$ interchanged by $f(z)$, we conclude that

$$f \in N_{n,\delta}(g;f).$$

This completes the proof of Theorem 2. □

Corollary 3 ([3]). *If $f \in \mathcal{K}_{\Omega}(p,\lambda,\mu,2,A,B)$, then*

$$\mathcal{K}_{\Omega}(p,\lambda,\mu,2,A,B) \subset N_{n,\delta}(g;f)$$

where $g(z)$ is given by (1.14) for $m = 2$ and δ is given by

$$\delta := \frac{p(B-A)(\lambda p - \lambda + 1)(n+p)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) - \lambda + 1]}\left[1 + \frac{(p+\mu)(p+\mu+1)}{n+p+\mu}\right].$$

Corollary 4. *If $f \in \mathcal{K}_{\Omega}(p,\lambda,\mu,3,A,B)$, then*

$$\mathcal{K}_{\Omega}(p,\lambda,\mu,3,A,B) \subset N_{n,\delta}(g;f)$$

where $g(z)$ is given by (1.14) for $m = 3$ and δ is given by

$$\delta := \frac{p(B-A)(\lambda p - \lambda + 1)(n+p)}{[(n+p)(1+B) - p(1+A)][\lambda(n+p) - \lambda + 1]}\left[1 + \frac{(p+\mu)(p+\mu+1)(p+\mu+2)}{2(n+p+\mu)(n+p+\mu+1)}\right].$$

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