



On the Involute of the Cubic Bezier Curve By Using Matrix Representation in \mathbb{E}^3

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Abstract. In this study we have examined, involute of the cubic Bezier curve based on the control points with matrix form in \mathbb{E}^3 . Frenet vector fields and also curvatures of involute of the cubic Bezier curve are examined based on the Frenet apparatus of the first cubic Bezier curve in \mathbb{E}^3 .

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1. Introduction and Preliminaries

In 1962 Bézier curves was studied by the French engineer Pierre Bézier, who used them to design automobile bodies. But the study of these curves was first developed in 1959 by mathematician Paul de Casteljau using de Casteljau's algorithm, a numerically stable method to evaluate Bézier curves. A Bézier curve is frequently used in computer graphics and related fields, in vector graphics, used in animation as a tool to control motion. For more detail using computer graphics see in [8]. In [2] some properties of Bezier curves are examined. To guarantee smoothness, the control point at which two curves meet must be on the line between the two control points on either side. In animation applications, such as Adobe Flash and Synfig, Bézier curves are used to outline, for example, movement. Users outline the wanted path in Bézier curves, and the application creates the needed frames for the object to move along the path. For 3D animation Bézier curves are often used to define 3D paths as well as 2D curves for keyframe interpolation. We have been motivated by the following studies. First Bezier-curves with curvature and torsion continuity has been examined in [5]. Also in [10] Bezier curves and surfaces has been given. In [3] planar Bezier curves and Bishop Frame of Bezier Curves are examined, respectively. Recently equivalence conditions of control points and application to planar Bezier curves have been examined in [6]. In this study we will define and work on Frenet apparatus of Bézier curves in \mathbb{E}^3 . So we need the derivatives of them. Recently Bezier-Like curves has been defined and cubic Bezier curves transitions have been studied in [7]. Also in [9] designing the ruled surface are examined as a new approach.

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Theorem 1. *The set, whose elements are Frenet vector fields and the curvatures of a curve $\alpha(t) \subset IE^3$, is called Frenet apparatus of the curves. Let $\alpha(t)$ be the curve, with $\eta = \|\alpha'(t)\| \neq 1$ and Frenet apparatus are $\{T(t), N(t), B(t), \kappa(t), \tau(t)\}$. Frenet vector fields are given for a non arc-lengthed curve*

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}, N(t) = B(t) \wedge T(t), B(t) = \frac{\alpha'(t) \wedge \alpha''(t)}{\|\alpha'(t) \wedge \alpha''(t)\|}$$

where curvature functions are defined by

$$\kappa(t) = \frac{\|\alpha'(t) \wedge \alpha''(t)\|}{\|\alpha'(t)\|^3}, \tau(t) = \frac{\langle \alpha'(t) \wedge \alpha''(t), \alpha'''(t) \rangle}{\|\alpha'(t) \wedge \alpha''(t)\|^2}.$$

Also Frenet formulae are well known as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \eta\kappa & 0 \\ -\eta\kappa & 0 & \eta\tau \\ 0 & -\eta\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, [4].$$

Theorem 2. *The Frenet-Serret vectors fields of the involute $\alpha^* = \alpha(t) + \lambda(t)T(t)$, which is not an arclengthed curve with $\|\alpha'\| = \eta \neq 1$, based on the its evolute curve α are*

$$T^* = N, N^* = \frac{-\kappa T + \tau B}{(\kappa^2 + \tau^2)^{\frac{1}{2}}}, B^* = \frac{\tau T + \kappa B}{(\kappa^2 + \tau^2)^{\frac{1}{2}}}.$$

The first and the second curvatures of involute α^* , are

$$\kappa^* = \frac{\sqrt{\kappa^2 + \tau^2}}{(c - \eta t)\kappa}, \tau^* = \frac{-\tau^2 \left(\frac{\kappa}{\tau}\right)'}{(c - \eta t)\kappa(\kappa^2 + \tau^2)} \tag{1}$$

respectively, where $\frac{dt}{ds^*} = \frac{1}{c - \eta t}$, [4].

Generally Bézier curve can be defined by $n + 1$ control points P_0, P_1, \dots, P_n with the parametrization

$$B(t) = \sum_{i=0}^n \binom{n}{i} t^i (1 - t)^{n-i} (t) [P_i]. \tag{2}$$

In this study we will define and work on cubic Bézier curves which are defined in E^3 . For more detail see [1].

Definition 1. *A cubic Bézier curve is a special Bézier curve has only four points P_0, P_1, P_2 and P_3 , with the parametrization*

$$B(t) = \sum_{i=0}^3 \binom{3}{i} t^i (1 - t)^{3-i} (t) [P_i], \tag{3}$$

$$B(t) = (1 - t)^3 P_0 + 3t(1 - t)^2 P_1 + 3t^2(1 - t) P_2 + t^3 P_3.$$

The matrix form of the cubic Bézier curve with control points P_0, P_1, P_2, P_3 , is

$$\alpha(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

Also using the derivatives of a cubic Bézier curve Frenet apparatus $\{T(t), N(t), B(t), \kappa(t), \tau(t)\}$ have already been given in [1] as in the following theorems by using matrix representation. For more detail see in [1].

Theorem 3. *The first derivative of a cubic Bézier curve by using matrix representation is*

$$\alpha'(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix} \tag{4}$$

with the control points

$$Q_0 = 3(P_1 - P_0) = (x_0, y_0, z_0), Q_1 = 3(P_2 - P_1) = (x_1, y_1, z_1) Q_2 = 3(P_3 - P_2) = (x_2, y_2, z_2).$$

Theorem 4. *The second derivative of a cubic Bézier curve by using matrix representation is*

$$\alpha''(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \end{bmatrix} \tag{5}$$

with the control points

$$\begin{aligned} R_0 &= 6(P_2 - 2P_1 + P_0) = 6(x_1 - x_0, y_1 - y_0, z_1 - z_0), \\ R_1 &= 6(P_3 - 2P_2 + P_1) = 6(x_2 - x_1, y_2 - y_1, z_2 - z_1). \end{aligned}$$

Theorem 5. *The third derivative of a cubic Bézier curve by using matrix representation is*

$$\alpha'''(t) = [R_0 R_1] \tag{6}$$

with the control points

$$[R_0 R_1] = R_1 - R_0 = 2[Q_1 Q_2] - 2[Q_0 Q_1] = 6(P_3 - 3P_2 + 3P_1 - P_0).$$

1.1. Frenet apparatus of a cubic Bézier curve

Frenet apparatus $\{T(t), N(t), B(t), \kappa(t), \tau(t)\}$ of a cubic Bézier curve have already been given in [1] as in the following theorems by using the matrix representation.

Theorem 6. *Tangent vector field of a cubic Bézier curve by using the matrix representation is*

$$T(t) = \frac{1}{\eta} \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$$

where $\eta = \|\alpha'\|$.

Theorem 7. Binormal vector field of a cubic Bezier curve by using the matrix representation is

$$B(t) = \frac{6}{m} \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

where $\|\alpha' \wedge \alpha''\| = m$

$$\begin{aligned} b_{11} &= y_0(z_1 - z_2) + y_1(z_2 - z_0) + y_2(z_0 - z_1) \\ b_{12} &= -x_0(z_1 - z_2) - x_1(z_2 - z_0) - x_2(z_0 - z_1) \\ b_{13} &= x_0(y_1 - y_2) + x_1(y_2 - y_0) + x_2(y_0 - y_1) \\ b_{21} &= 2y_1z_0 + y_0z_2 - 2y_0z_1 - y_2z_0 \\ b_{22} &= 2x_0z_1 - 2x_1z_0 - x_0z_2 + x_2z_0 \\ b_{23} &= 2x_1y_0 - 2x_0y_1 + x_0y_2 - x_2y_0 \\ b_{31} &= y_0z_1 - y_1z_0 \\ b_{32} &= x_1z_0 - x_0z_1 \\ b_{33} &= x_0y_1 - x_1y_0. \end{aligned}$$

Theorem 8. Normal vector field of a cubic Bezier curve by using the matrix representation is

$$N(t) = \frac{6}{\eta m} \begin{bmatrix} t^4 & t^3 & t^2 & t^1 & 1 \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \\ n_{41} & n_{41} & n_{43} \\ n_{51} & n_{51} & n_{53} \end{bmatrix}$$

where

$$\begin{aligned} n_{11} &= b_{12}d_{13} - b_{13}d_{12} \\ n_{21} &= b_{12}d_{23} - b_{13}d_{22} + b_{22}d_{13} - b_{23}d_{12} \\ n_{31} &= b_{12}d_{33} - b_{13}d_{32} + b_{22}d_{23} - b_{23}d_{22} + b_{32}d_{13} - b_{33}d_{12} \\ n_{41} &= b_{22}d_{33} - b_{23}d_{32} + b_{32}d_{23} - b_{33}d_{22} \\ n_{51} &= b_{32}d_{33} - b_{33}d_{32} \\ n_{12} &= b_{11}d_{13} - b_{13}d_{11} \\ n_{22} &= -b_{11}d_{23} - b_{21}d_{13} + b_{13}d_{21} + b_{23}d_{11} \\ n_{32} &= b_{23}d_{21} + b_{33}d_{11} - b_{11}d_{33} - b_{21}d_{23} + b_{13}d_{31} - b_{31}d_{13} \\ n_{42} &= -b_{21}d_{33} - b_{31}d_{23} + b_{23}d_{31} + b_{33}d_{21} \\ n_{52} &= -b_{31}d_{33} + b_{33}d_{31} \\ n_{13} &= b_{11}d_{12} - b_{12}d_{11} \\ n_{23} &= b_{11}d_{22} - b_{12}d_{21} + b_{21}d_{12} - b_{22}d_{11} \\ n_{33} &= b_{11}d_{32} - b_{12}d_{31} + b_{21}d_{22} - b_{22}d_{21} + b_{31}d_{12} - b_{32}d_{11} \\ n_{43} &= b_{21}d_{32} - b_{22}d_{31} + b_{31}d_{22} - b_{32}d_{21} \\ n_{53} &= b_{31}d_{32} - b_{32}d_{31}. \end{aligned}$$

Theorem 9. First and second curvatures of a cubic Bezier curve by using the matrix

representation are

$$\kappa(t) = \frac{6}{\eta^3} \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} b_{11}^2 + b_{12}^2 + b_{13}^2 \\ 2b_{11}b_{21} + 2b_{12}b_{22} + 2b_{13}b_{23} \\ 2b_{11}b_{31} + 2b_{12}b_{32} + 2b_{13}b_{33} + b_{21}^2 + b_{22}^2 + b_{23}^2 \\ 2b_{21}b_{31} + 2b_{22}b_{32} + 2b_{23}b_{33} \\ b_{31}^2 + b_{32}^2 + b_{33}^2 \end{bmatrix},$$

$$\tau(t) = \frac{x_0y_1z_2 - x_0y_2z_1 - x_1y_0z_2 + x_1y_2z_0 + x_2y_0z_1 - x_2y_1z_0}{\|\alpha'(t) \wedge \alpha''(t)\|^2}.$$

2. Involute of cubic Bezier curve

Definition 2. If the curve α^* which lies on the tangent surface intersect the tangent lines orthogonally is called an involute of α . If a curve α^* is an involute of α , then by definition α which is not an arclengthed curve is an evolute of α^* . Hence given α , its evolutes are the curves whose tangent lines intersect α orthogonally. Let the quantities $\{T^*, N^*, B^*, \kappa^*, \tau^*\}$ be collectively Frenet-Serret apparatus of the curve α^* which is not an arclengthed curve with $\|\alpha'\| = \eta \neq 1$, [4]. The equation of involute of the curve α has the following parametrization;

$$\alpha^*(t) = \alpha(t) + \lambda(t) T(t). \tag{7}$$

Also since $\lambda = c - \eta t$ it can be written as in the following parametrization

$$\alpha^*(t) = \alpha(t) + \frac{(c - \eta t)}{\eta} \alpha'(t).$$

Theorem 10. The involute of a cubic Bezier curve has the matrix form based on the control points P_0, P_1, P_2 and P_3 of any cubic Bezier curve

$$\alpha^*(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 - 3\mu & -6 + 9\mu & 3 - 9\mu & 3\mu \\ -3 + 6\mu & 3 - 12\mu & 6\mu & 0 \\ 1 + 3\mu & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

with $\mu = \frac{c - \eta t}{\eta}$.

Proof. Lets $\mu = \frac{(c - \eta t)}{\eta}$, since $\alpha^* = \alpha(t) + \mu \alpha'(t)$

$$\alpha^* = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} + \mu \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}$$

and $Q_0 = 3(P_1 - P_0), Q_1 = 3(P_2 - P_1), Q_2 = 3(P_3 - P_2)$ it can be written in matrix form as in

$$\alpha^*(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ (3-3\mu) & (-6+9\mu) & (3-9\mu) & 3\mu \\ (-3+6\mu) & (3-12\mu) & 6\mu & 0 \\ (1+3\mu) & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

we have its matrix product form as

$$\alpha^*(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 3P_1 - P_0 - 3P_2 + P_3 \\ P_1(9\mu - 6) - P_2(9\mu - 3) - P_0(3\mu - 3) + 3\mu P_3 \\ P_0(6\mu - 3) - P_1(12\mu - 3) + 6\mu P_2 \\ P_0(3\mu + 1) - 3P_1 \end{bmatrix}. \tag{8}$$

Theorem 11. *The control points of the involute of any cubic Bezier curve with constant speed, based on the control points of cubic Bezier curve, as in the following way*

$$\begin{aligned} I_0 &= 3\frac{c}{\eta}P_1 - P_0 \left(3\frac{c}{\eta} - 1\right), \\ I_1 &= 3\frac{c}{\eta}P_1 - P_0 \left(\frac{c}{\eta} - 1\right) - 2\frac{c}{\eta}P_2, \\ I_2 &= \left(6\frac{c}{\eta} + 2\right)P_1 - P_2 \left(7\frac{c}{\eta} + 1\right) + \frac{c}{\eta}P_3, \\ I_3 &= \left(3\frac{c}{\eta} - 2\right)P_3 - P_2 \left(15\frac{c}{\eta} - 3\right) + 12\frac{c}{\eta}P_1. \end{aligned}$$

Proof. Let $I_0, I_1, I_2,$ and I_3 be control points of involute α^* , so we can write

$$\alpha^*(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \end{bmatrix}. \tag{9}$$

From the equality of the left sides of 8 and 9, we have

$$\begin{aligned} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \end{bmatrix} &= \begin{bmatrix} 2P_0 - 6P_1 + 6P_2 - 2P_3 \\ \left(-3 - \frac{3c}{\eta}\right)P_0 + \left(6 + \frac{9c}{\eta}\right)P_1 - \left(3 + \frac{9c}{\eta}\right)P_2 + \frac{3c}{\eta}P_3 \\ \frac{6c}{\eta}P_0 - \frac{6c}{\eta}P_2 \\ \left(1 - \frac{3c}{\eta}\right)P_0 + \frac{3c}{\eta}P_1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -6 & 6 & -2 \\ \left(-3 - \frac{3c}{\eta}\right) & \left(6 + \frac{9c}{\eta}\right) & -\left(3 + \frac{9c}{\eta}\right) & +\frac{3c}{\eta} \\ \frac{6c}{\eta} & 0 & -\frac{6c}{\eta} & 0 \\ \left(1 - \frac{3c}{\eta}\right) & \frac{3c}{\eta} & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \end{aligned}$$

using the inverse matrix

$$\begin{bmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -6 & 6 & -2 \\ \left(-3 - \frac{3c}{\eta}\right) & \left(6 + \frac{9c}{\eta}\right) & -\left(3 + \frac{9c}{\eta}\right) & +\frac{3c}{\eta} \\ \frac{6c}{\eta} & 0 & -\frac{6c}{\eta} & 0 \\ \left(1 - \frac{3c}{\eta}\right) & \frac{3c}{\eta} & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}. \tag{10}$$

Theorem 12. *The control points of the involute of any cubic Bezier curve, under the condition $\frac{c}{\eta} - t = \mu = \text{constant}$, can be given,*

$$\begin{aligned} I_0 &= 3\mu P_1 - (3\mu - 1) P_0, \\ I_1 &= 2\mu P_2 - \mu P_0 - P_1 (\mu - 1), \\ I_2 &= \mu P_3 - 2\mu P_1 + P_2 (\mu + 1), \\ I_3 &= (3\mu + 1) P_3 - 3\mu P_2. \end{aligned}$$

Proof. If $\frac{c}{\eta} - t = \mu$ is constant,

$$\begin{aligned} \alpha^*(t) &= [t^3 \ t^2 \ t \ 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \\ &+ \mu [t^2 \ t \ 1] \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3(P_1 - P_0) \\ 3(P_2 - P_1) \\ 3(P_3 - P_2) \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \end{bmatrix} &= \begin{bmatrix} -P_0 + 3P_1 - 3P_2 + P_3 \\ ((3 - 3\mu) P_0 + (-6 + 9\mu) P_1 + (3 - 9\mu) P_2 + 3\mu P_3) \\ ((-3 + 6\mu) P_0 + (3 - 12\mu) P_1 + 6\mu P_2) \\ (1 - 3\mu) P_0 + 3\mu P_1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 & -3 & 1 \\ (3 - 3\mu) & (-6 + 9\mu) & (3 - 9\mu) & 3\mu \\ (-3 + 6\mu) & (3 - 12\mu) & 6\mu & 0 \\ (1 - 3\mu) & 3\mu & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \end{aligned}$$

using the inverse matrix we can find the control points of the involute of any cubic Bezier curve with constant μ , based on the control points of cubic Bezier curve, as in the following way:

$$\begin{bmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 3 & -3 & 1 \\ (3 - 3\mu) & (-6 + 9\mu) & (3 - 9\mu) & 3\mu \\ (-3 + 6\mu) & (3 - 12\mu) & 6\mu & 0 \\ (1 - 3\mu) & 3\mu & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix},$$

$$\begin{bmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ (3-3\mu) & (-6+9\mu) & (3-9\mu) & 3\mu \\ (-3+6\mu) & (3-12\mu) & 6\mu & 0 \\ (1-3\mu) & 3\mu & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}. \tag{11}$$

2.1. Frenet apparatus of the involute curve of any cubic Bezier curve in E^3

Theorem 13. *Tangent vector field of involute curve of any cubic Bezier curve is*

$$T^* = \frac{6}{m\eta} \begin{bmatrix} t^4 & t^3 & t^2 & t^1 & 1 \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \\ n_{41} & n_{41} & n_{43} \\ n_{51} & n_{51} & n_{53} \end{bmatrix}.$$

Proof. We have already known that tangent vector field of involute curve T^* is linear dependent N , that is why $T^* = N$.

Theorem 14. *Normal vector field of involute α^* of any cubic Bezier curve in E^3 is*

$$N^* = \frac{\begin{bmatrix} t^2 & t & 1 \end{bmatrix}}{\eta \frac{(\kappa^2 + \tau^2)^{\frac{1}{2}}}{\kappa}} \begin{bmatrix} 2x_1 - x_0 - x_2 + \frac{6\eta\tau}{m\kappa}b_{11} & 2y_1 - y_0 - y_2 + \frac{6\eta\tau}{m\kappa}b_{12} & 2z_1 - z_0 - z_2 + \frac{6\eta\tau}{m\kappa}b_{13} \\ 2x_0 - 2x_1 + \frac{6\eta\tau}{m\kappa}b_{21} & 2y_0 - 2y_1 + \frac{6\eta\tau}{m\kappa}b_{22} & 2z_0 - 2z_1 + \frac{6\eta\tau}{m\kappa}b_{23} \\ \frac{6\eta\tau}{m\kappa}b_{31} - x_0 & \frac{6\eta\tau}{m\kappa}b_{32} - y_0 & \frac{6\eta\tau}{m\kappa}b_{33} - z_0 \end{bmatrix}.$$

Proof. Since $N^* = \frac{-\kappa T + \tau B}{(\kappa^2 + \tau^2)^{\frac{1}{2}}}$, we have

$$N^* = \frac{\frac{-\kappa}{\eta} \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} + \frac{\tau}{m} \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}{(\kappa^2 + \tau^2)^{\frac{1}{2}}}$$

$$N^* = \frac{\begin{bmatrix} t^2 & t & 1 \end{bmatrix}}{(\kappa^2 + \tau^2)^{\frac{1}{2}}} \left(\frac{-\kappa}{\eta} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} + \frac{6\tau}{m} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \right).$$

Hence it is easy to calculate the following result

$$N^* = \frac{\begin{bmatrix} t^2 & t & 1 \end{bmatrix}}{\eta \frac{(\kappa^2 + \tau^2)^{\frac{1}{2}}}{\kappa}} \begin{bmatrix} 2x_1 - x_0 - x_2 + \frac{6\eta\tau}{m\kappa}b_{11} & 2y_1 - y_0 - y_2 + \frac{6\eta\tau}{m\kappa}b_{12} & 2z_1 - z_0 - z_2 + \frac{6\eta\tau}{m\kappa}b_{13} \\ 2x_0 - 2x_1 + \frac{6\eta\tau}{m\kappa}b_{21} & 2y_0 - 2y_1 + \frac{6\eta\tau}{m\kappa}b_{22} & 2z_0 - 2z_1 + \frac{6\eta\tau}{m\kappa}b_{23} \\ \frac{6\eta\tau}{m\kappa}b_{31} - x_0 & \frac{6\eta\tau}{m\kappa}b_{32} - y_0 & \frac{6\eta\tau}{m\kappa}b_{33} - z_0 \end{bmatrix}.$$

Theorem 15. Binormal vector field of involute α^* of any cubic Bezier curve in \mathbf{E}^3 is

$$B^* = \frac{\begin{bmatrix} t^2 & t & 1 \end{bmatrix}}{\frac{(\kappa^2 + \tau^2)^{\frac{1}{2}}}{\tau} \eta} \begin{bmatrix} x_0 - 2x_1 + x_2 + \frac{6\eta\kappa}{m\tau} b_{11} & y_0 - 2y_1 + y_2 + \frac{6\eta\kappa}{m\tau} b_{12} & z_0 - 2z_1 + z_2 + \frac{6\eta\kappa}{m\tau} b_{13} \\ 2x_1 - 2x_0 + \frac{6\eta\kappa}{m\tau} b_{21} & 2y_1 - 2y_0 + \frac{6\eta\kappa}{m\tau} b_{22} & 2z_1 - 2z_0 + \frac{6\eta\kappa}{m\tau} b_{23} \\ x_0 + \frac{6\eta\kappa}{m\tau} b_{31} & y_0 + \frac{6\eta\kappa}{m\tau} b_{32} & z_0 + \frac{6\eta\kappa}{m\tau} b_{33} \end{bmatrix}.$$

Proof. Since $B^* = \frac{\tau T + \kappa B}{(\kappa^2 + \tau^2)^{\frac{1}{2}}}$,

$$B^* = \frac{\frac{\tau}{\eta} \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} + \frac{6\kappa}{m} \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}{(\kappa^2 + \tau^2)^{\frac{1}{2}}},$$

$$B^* = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \frac{\frac{\tau}{\eta} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} + \frac{6\kappa}{m} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}{(\kappa^2 + \tau^2)^{\frac{1}{2}}}.$$

Hence it is easy to give the proof.

2.2. The first and second curvature of involute α^*

Theorem 16. The first curvature of involute α^* of any cubic Bezier curve in \mathbf{E}^3 is

$$\kappa^* = \frac{1}{m^2(c - \eta t)} \sqrt{m^4 + \eta^6 (x_0 y_1 z_2 - x_0 y_2 z_1 - x_1 y_0 z_2 + x_1 y_2 z_0 + x_2 y_0 z_1 - x_2 y_1 z_0)^2}$$

Proof. Since $\kappa^* = \frac{\sqrt{\kappa^2 + \tau^2}}{(c - \eta t)\kappa}$, and $\kappa = \frac{m}{\eta^3}$

$$\kappa^* = \frac{\sqrt{\frac{m^2}{\eta^6} + \frac{(x_0 y_1 z_2 - x_0 y_2 z_1 - x_1 y_0 z_2 + x_1 y_2 z_0 + x_2 y_0 z_1 - x_2 y_1 z_0)^2}{m^2}}}{(c - \eta t) \frac{m}{\eta^3}}$$

$(c - \eta t)\kappa > 0, \kappa \neq 0$. It is trivial.

Theorem 17. The second curvature of involute α^* of any cubic Bezier curve in \mathbf{E}^3 is

$$\tau^* = \frac{\eta^3}{m^3} \left(\frac{\kappa - \kappa' (x_0 y_1 z_2 - x_0 y_2 z_1 - x_1 y_0 z_2 + x_1 y_2 z_0 + x_2 y_0 z_1 - x_2 y_1 z_0)}{(c - \eta t) (\kappa^2 + \tau^2)} \right).$$

Proof.

Considering the $\tau^* = \frac{\kappa\tau' - \kappa'\tau}{(c - \eta t)\kappa(\kappa^2 + \tau^2)}$ equation, we have

$$\tau^* = \frac{\kappa - \kappa' (x_0 y_1 z_2 - x_0 y_2 z_1 - x_1 y_0 z_2 + x_1 y_2 z_0 + x_2 y_0 z_1 - x_2 y_1 z_0)}{m^2(c - \eta t) \frac{m}{\eta^3} (\kappa^2 + \tau^2)}.$$

Example 1. Find the involute of the cubic Bezier curve with control points $P_0 = (1, 2, 3)$, $P_1 = (1, 1, 1)$, $P_2 = (2, 1, 3)$, and $P_3 = (1, -1, 0)$. The cubic Bezier curve has the following matrix representation

$$\alpha(t) = [t^3 \ t^2 \ t \ 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}. \quad (12)$$

The involute α^* of the cubic Bezier curve α , has the following matrix representation

$$\alpha^*(t) = [t^3 \ t^2 \ t \ 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3-3\mu & -6+9\mu & 3-9\mu & 3\mu \\ -3+6\mu & 3-12\mu & 6\mu & 0 \\ 1+3\mu & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}. \quad (13)$$

Using control points as

$$\alpha^*(t) = [t^3 \ t^2 \ t \ 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3-3\mu & -6+9\mu & 3-9\mu & 3\mu \\ -3+6\mu & 3-12\mu & 6\mu & 0 \\ 1+3\mu & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

we have

$$\alpha^*(t) = \begin{pmatrix} 3\mu + 6t\mu - 9t^2\mu + 3t^2 - 3t^3 - 2, 6\mu - 3t + 6t\mu - 9t^2\mu + 3t^2 - 3t^3, \\ 24t\mu - 19\mu - 6t - 27t^2\mu + 12t^2 - 9t^3 \end{pmatrix}.$$

Also under the condition constant $c = \eta = \|\alpha'\|$ and $\mu = (1-t)$, we can find the special involute of α as in the following way

$$\alpha^*(t) = [t^3 \ t^2 \ t \ 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3-3(1-t) & -6+9(1-t) & 3-9(1-t) & 3(1-t) \\ -3+6(1-t) & 3-12(1-t) & 6(1-t) & 0 \\ 1+3(1-t) & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\alpha^*(t) = [t^3 \ t^2 \ t \ 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3t & 3-9t & 9t-6 & 3-3t \\ 3-6t & 12t-9 & 6-6t & 0 \\ 4-3t & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

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