

Rotating cosmological cylindrical wormholes in GR and TEGR sourced by anisotropic fluids

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ABSTRACT

Given an anisotropic fluid source, we determine in closed forms, upon solving the field equations of general relativity (GR) and teleparallel gravity (TEGR) coupled to a cosmological constant, cylindrically symmetric four-dimensional cosmological rotating wormholes, satisfying all local energy conditions, and cosmological rotating solutions with two axes of symmetry at finite proper distance. These solutions have the property that their angular velocity is proportional to the cosmological constant.

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1. Introduction

The teleparallel equivalent of general relativity (TEGR) is an equivalent formulation of Einstein's general relativity (GR), in that, the formulation ensures equivalence of the field equations as well as of the test particle equations of motion. For a review see the paper by Maluf [1] where an account of the history of teleparallel theories of gravity is given. The two theories, TEGR and GR, use different connections, the curvature-less Weitzenböck connection [2] and the torsion-less Levi-Civita connection [3], respectively. The Weitzenböck connection has the property that it allows for a definition of a condition for absolute parallelism in space-time [1], hence the given name of "teleparallel". The tensors and invariants associated to TEGR exhibit no curvature but torsion only, that is, the information concerning the gravitational field effects are encoded in the torsion tensor instead of the Riemann tensor, as it is the case in GR.

Despite their equivalence, the two theories are conceptually different. All physical features and results known in GR are also described in the TEGR. The converse is not true: The TEGR approach allows the consideration of additional concepts and definitions. In the TEGR one meets tensors with three indices some of which are behind the definition of the gravitational energy-momentum which is consistent with the field equations [4,5]. The concept and definition of the gravitational angular momentum are also introduced in the TEGR [1]. From the point of view of test particles, there is no notion of geodesics in the TEGR but only force equations and the source of force is torsion, more precisely the force is sum of a torsion-tensor component times two components of the velocity vector [6]. In the TEGR it is

also possible to introduce the concept of inertia-free frames [6] and to split particle dynamical effects into distinct gravitational effects and purely inertial effects, that is, it is possible to separate inertia from gravity [7]. Another example of conceptual difference between the two theories is provided in this work following Eq. (17).

We have already mentioned some motivations for the investigation of the TEGR: Introduction of new concepts and unambiguous definitions such as that of the gravitational energy-momentum and other definitions [1] and the clear distinction of purely inertial effects from gravitational ones. Other motivations for the TEGR is that the theory admits some extensions by adding quadratic and higher-order torsion terms to the action making it a good cosmological dark energy model without truly adding exotic matter to the cosmological field equations. Said otherwise, these extra terms added to the action, the so-called $f(T)$ gravity, have their counterparts in the cosmological field equations playing the "role" of exotic matter [1,8]. The extended TEGR may provide a theoretical model to the late time universe acceleration problem [9].

Some rotating cylinders in general relativity sourced by anisotropic fluids and $\Lambda = 0$ have been determined in [10–12]. The purpose of this work is to consider the theories of GR and TEGR coupled to a cosmological constant and determine cylindrically symmetric rotating wormholes and other solutions sourced by anisotropic fluids.

Determining rotating and static solutions around an infinite axis is still a reviving topic. In GR there is a set of rotating cylindrically symmetric perfect fluid solutions which may be appropriate as matched interiors. Among the known solutions in GR we find the rotating dust of Vishveshwara and Winicour [13], the perfect fluid sources with non-zero pressure of da Silva et al. [14],

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Davidson [15,16] and Ivanov [17], and the family of Krasinski [18]. The solutions that will be constructed in this work share some physical and geometrical properties with the solutions known in the literature and have some other new properties. Among the new properties we mention that their angular velocity is proportional to the cosmological constant.

Investigating wormhole solutions is another, rather many-fold, reviving topic. First of all, wormholes are special types of solutions to the field equations of gravity theories which contain two distant asymptotic regions or sheets, with a throat connecting the two and providing a shortcut for long journeys from one asymptotic region to another [19,20]. Their observation has not been confirmed yet but they may be very well lurking in the universe. In Ref. [21] it was shown that the diameter of the shadow of type I supermassive wormhole overlaps with that of the black hole candidate at the center of the Milky Way. This shows that the existing up-to-date millimeter-wavelength very long baseline interferometry facilities do not lead to differentiate the supermassive black hole candidate at the center of the Milky Way from a possible type I supermassive wormhole. Very recently, it was shown that the active galactic nuclei exhibit wormhole behaviors rather than supermassive black holes due to their gamma radiation resulting from collision of accreting flows [22]. Said otherwise, wormholes may be spotted in the sky upon detecting the gigantesque display of gamma rays that results from the collision of matter coming out of one mouth of the wormhole with infalling matter [22]. A nearly similar conclusion was made in Ref. [21]: "Other signals from the galaxy, as the motion of orbiting hot spots, may lead to draw a conclusion concerning the nature of the candidate".

The interest to wormholes goes back to 1935. Einstein and N. Rosen were the first who described how two distant regions of spacetime can be joined together, creating a bridge between them. Most of the known solutions, and their list is too large to be cited here, are endowed with spherical symmetry, until recently a couple of wormhole solutions endowed with cylindrical symmetry were determined (see [23] are references therein). The construction of analytical wormhole solutions, with spherical or cylindrical symmetry, provides the scientific community with theoretical tools for further investigations (calculations of shadow, quasi-normal modes, quasi-periodic oscillations, etc.), as was done in [21,24,25], and for computer simulations, as was done in [26,27] to study the collisional processes in the geometry of rotating wormholes.

In Section 2 we introduce the mathematical tools needed for the TEGR along with the necessary field equations. In Section 3 we reduce the field equations. In Section 4 we restrict ourselves to anisotropic fluids with $p_r = \omega_r \rho$, $p_\phi = \omega_\phi \rho$ and $p_z = \omega_z \rho$, where the equation-of-state (EoS) parameters (ω_r , ω_ϕ , ω_z) are constants constrained by $-1 \leq \omega_r \leq 1$, $-1 \leq \omega_\phi \leq 1$ and $-1 \leq \omega_z \leq 1$. Section 5 is devoted to the construction of rotating wormholes and Section 6 is devoted to the discussion of their physical and geometrical properties. In Section 7 we provide cosmological rotating solutions with two axes of symmetry at finite proper distance. Our final conclusions are given in Section 8. An Appendix section has been added to provide some useful formulas pertaining to Section 5.

2. The teleparallel equivalent of general relativity

In the TEGR [28] the vielbein vector fields $e_a = e_a^\mu \partial_\mu$ are taken as fundamental variables instead of the metric $g_{\mu\nu}$, related to each other by

$$g_{\mu\nu} = \eta_{ab} e_a^\mu e_b^\nu, \quad g^{\mu\nu} = \eta^{ab} e_a^\mu e_b^\nu, \quad e = \sqrt{|g|}, \quad (1)$$

with $\eta_{ab} = \text{diag}(+1, -1, -1, -1)$ being the metric of the 4-dimensional Minkowski spacetime and $e \equiv |\det(e^a_\mu)| = \sqrt{|g|}$. In

this work the tetrad indices, $a, b, c \dots, k, l$, and Greek coordinate indices run from 0 to 3.

The field equation in teleparallel gravity in the presence of matter fields take the form [29,30]

$$I_\mu{}^\nu := -\delta_\mu^\nu \frac{f}{2} + 2 \left[e^{-1} e^\alpha{}_\mu \partial_\rho (e e_a^\alpha S_a{}^{\rho\nu}) - T^\alpha{}_{\lambda\mu} S_\alpha{}^{\nu\lambda} \right] = -\kappa T_{(\text{mat})\mu}{}^\nu, \quad (2)$$

where κ is the gravitational constant,

$$f(T) = T + 2\Lambda, \quad (3)$$

and $T_{(\text{mat})\mu}{}^\nu$ is the matter stress-energy tensor (SET) which we assume to be that of an anisotropic fluid of the form

$$T_{(\text{mat})\mu}{}^\nu = \rho u_\mu u^\nu + \sum_{a=1}^3 p_a e_{a\mu} e_a{}^\nu = (\rho + p_1) u_\mu u^\nu - p_1 \delta_\mu^\nu + (p_2 - p_1) e_{2\mu} e_2{}^\nu + (p_3 - p_1) e_{3\mu} e_3{}^\nu. \quad (4)$$

Here we have chosen $e_0{}^\nu = u^\nu$ to be the four-velocity vector of the fluid. The remaining quantities used in (2), including the torsion T , are defined by¹

$$T^\alpha{}_{\mu\nu} = e_b{}^\alpha (\partial_\mu e_b{}^\nu - \partial_\nu e_b{}^\mu), \\ K_{\alpha\mu\nu} = \frac{1}{2} (T_{\mu\alpha\nu} + T_{\nu\alpha\mu} - T_{\alpha\mu\nu}), \\ S^{\alpha\mu\nu} = \frac{1}{2} (K^{\mu\nu\alpha} - g^{\alpha\nu} T^\sigma{}_\mu{}^\sigma + g^{\alpha\mu} T^{\sigma\nu}{}_\sigma), \\ T = T_{\alpha\mu\nu} S^{\alpha\mu\nu}. \quad (5)$$

In the definition of $T^\alpha{}_{\mu\nu}$ the connection $\omega^a{}_{b\mu}$ has been set equal to zero as this is always possible in teleparallel gravity [31].

In cylindrical coordinates ($x^0 = t$, $x^1 = r$, $x^2 = \phi$, $x^3 = z$), we introduce the following non-diagonal vielbein to describe rotating solutions

$$(e^a{}_\mu) = \begin{pmatrix} e^{\gamma(r)} & 0 & -e^{-\gamma(r)} \Omega(r) & 0 \\ 0 & e^{\alpha(r)} & 0 & 0 \\ 0 & 0 & e^{\beta(r)} & 0 \\ 0 & 0 & 0 & e^{\mu(r)} \end{pmatrix}, \quad (6)$$

resulting in the metric

$$ds^2 = e^{2\gamma(r)} [dt - \Omega(r) e^{-2\gamma(r)} d\phi]^2 - e^{2\alpha(r)} dr^2 - e^{2\beta(r)} d\phi^2 - e^{2\mu(r)} dz^2. \quad (7)$$

Using all that in (5) we obtain

$$T = \frac{1}{2} e^{-2(\alpha+\beta+\gamma)} (\Omega'^2 - 2\Omega\gamma')^2 + 2e^{-2\alpha} (\beta'\gamma' + \beta'\mu' + \gamma'\mu'). \quad (8)$$

The nonvanishing components of the SET (4) are

$$T_{(\text{mat})t}{}^t = \rho, \quad T_{(\text{mat})r}{}^r = -p_r, \quad T_{(\text{mat})\phi}{}^\phi = -p_\phi, \\ T_{(\text{mat})z}{}^z = -p_z, \quad T_{(\text{mat})\phi}{}^t = -\Omega(\rho + p_\phi) e^{-2\gamma}, \quad (9)$$

where we have set $p_1 = p_r$, $p_2 = p_\phi$, $p_3 = p_z$.

It has become customary to introduce the vortex $\omega(r)$, which is the norm of the curl of the tetrad $e_a{}^\mu$ [32] (see also [12,33–35]).

¹ $S^{\alpha\mu\nu}$ may be given in a more compact form as:

$$S^{\alpha\mu\nu} = \frac{1}{4} (T^{\nu\mu\alpha} + T^{\alpha\mu\nu} - T^{\mu\nu\alpha}) - \frac{1}{2} g^{\alpha\nu} T^\sigma{}_\mu{}^\sigma + \frac{1}{2} g^{\alpha\mu} T^{\sigma\nu}{}_\sigma.$$

This is related to $\Omega(r)$ by²

$$\Omega(r) := 2e^{2\gamma(r)} \int^r e^{\alpha(x)+\beta(x)-\gamma(x)} \omega(x) dx, \quad (10)$$

yielding

$$T = 2\omega^2 + 2e^{-2\alpha}(\beta'\gamma' + \beta'\mu' + \gamma'\mu'). \quad (11)$$

One may bring the field Eqs. (2) to³

$$G_{\mu}^{\nu} = -\kappa\tau_{\mu}^{\nu} \quad (12)$$

where G_{μ}^{ν} is the Einstein tensor and τ_{μ}^{ν} is the total SET including the cosmological constant and is defined by

$$\begin{aligned} \tau_t^t &= \rho - \frac{\Lambda}{\kappa}, & \tau_r^r &= -p_r - \frac{\Lambda}{\kappa}, \\ \tau_{\phi}^{\phi} &= -p_{\phi} - \frac{\Lambda}{\kappa}, & \tau_z^z &= -p_z - \frac{\Lambda}{\kappa}, \\ \tau_{\phi}^t &= -\Omega(\rho + p_{\phi})e^{-2\gamma}. \end{aligned} \quad (13)$$

Since $\nabla_{\nu}G_{\mu}^{\nu} \equiv 0$, one must have

$$\nabla_{\nu}\tau_{\mu}^{\nu} = 0. \quad (14)$$

The solutions we will derive in this work will satisfy the field equations of GR (12) and of TEGR (2). For TEGR, the solutions are constrained by (11).

3. Reducing the field equations

Given that $T_{(\text{mat})t}^{\phi} = 0$ and $\tau_t^{\phi} = 0$, the line $I_t^{\phi} = 0$ in (2), and the component t^{ϕ} of Eq. (12), reduce upon using (9) and (13) to

$$I_t^{\phi} = G_t^{\phi} = -e^{\gamma-\alpha-\beta}[\omega(2\gamma' + \mu') + \omega'] = 0,$$

yielding

$$\omega(r) = \omega_0 e^{-2\gamma-\mu}, \quad (15)$$

where ω_0 is a constant of integration. Using this in (10) we obtain

$$\Omega(r) = 2\omega_0 e^{2\gamma(r)} \int^r e^{\alpha(x)+\beta(x)-3\gamma(x)-\mu(x)} dx. \quad (16)$$

The expression of G_r^r is just half that of T (11), $G_r^r = \omega^2 + e^{-2\alpha}(\beta'\gamma' + \beta'\mu' + \gamma'\mu') = T/2$, and this implies using (12) and (13) that p_r is given by

$$p_r = \frac{T}{2\kappa} - \frac{\Lambda}{\kappa}, \quad (17)$$

where the last term proportional to Λ is the radial pressure due to the cosmological constant and the first term, $T/(2\kappa)$, is the radial pressure generated by a constant torsion. Thus, in the TEGR, a 'nonvanishing' torsion generates a nonvanishing radial pressure $T/(2\kappa)$ while in GR the relation (17) is written as $G_r^r = \kappa p_r + \Lambda$ and it merely expresses the fact that the geometric entity G_r^r is proportional to the sum of the pressures p_r and Λ/κ . This is another example of conceptual difference between the two theories that we mentioned in the Introduction. This is at the level of the field equations, where the torsion appears as a force while

² The curl ω^{μ} of e_a^{μ} is

$$\omega^{\mu} := \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} e_{\alpha\nu} e^{\alpha}_{\rho;\sigma},$$

yielding $\omega^t = \omega^r = \omega^{\phi} = 0$, $\omega^z = -e^{\gamma-\alpha-\beta-\mu}(\Omega e^{-2\gamma}\gamma')/2$ and $\omega = \sqrt{\omega^{\mu}\omega_{\mu}} = e^{\gamma-\alpha-\beta}(\Omega e^{-2\gamma}\gamma')/2$.

³ In this work

$$R^{\alpha}_{\beta\mu\nu} := \partial_{\nu}\Gamma^{\alpha}_{\beta\mu} - \partial_{\mu}\Gamma^{\alpha}_{\beta\nu} + \Gamma^{\alpha}_{\eta\nu}\Gamma^{\eta}_{\beta\mu} - \Gamma^{\alpha}_{\eta\mu}\Gamma^{\eta}_{\beta\nu}.$$

its counterpart in GR, the curvature scalar \mathcal{R} , does not assume a dynamical role in the field equations.

Substituting (13) into (14) we obtain

$$-4p'_r - (\beta' + \gamma' + \mu')p_r - \gamma'\rho + \beta'p_{\phi} + \mu'p_z = 0. \quad (18)$$

4. Anisotropic fluids

We consider the simple case where $p_r = \omega_r\rho$, $p_{\phi} = \omega_{\phi}\rho$ and $p_z = \omega_z\rho$, with the EoS parameters (ω_r , ω_{ϕ} , ω_z) being constants generally constrained by

$$-1 \leq \omega_r \leq 1, \quad -1 \leq \omega_{\phi} \leq 1, \quad -1 \leq \omega_z \leq 1. \quad (19)$$

The differential equation (18) becomes

$$(\omega_r - \omega_{\phi})\beta' + (1 + \omega_r)\gamma' + (\omega_r - \omega_z)\mu' + 4\omega_r \frac{\rho'}{\rho} = 0, \quad (20)$$

resulting in

$$\rho = \rho_0 \exp\left[\frac{(\omega_{\phi} - \omega_r)\beta - (1 + \omega_r)\gamma + (\omega_z - \omega_r)\mu}{4\omega_r}\right], \quad (21)$$

where ρ_0 is a constant of integration. Using this in (17) we can evaluate the torsion from

$$T = 2(\kappa\omega_r\rho + \Lambda). \quad (22)$$

Since the radial coordinate r can be changed at will by a coordinate transformation $r \rightarrow \bar{r}$, from now on we fix the coordinate gauge to be

$$\alpha = \beta + \gamma + \mu. \quad (23)$$

In this gauge the independent field equations emanating from (12)

$$R_{\mu}^{\nu} = -\kappa[\tau_{\mu}^{\nu} - (1/2)\delta_{\mu}^{\nu}\tau_{\sigma}^{\sigma}], \quad (24)$$

reduce to

$$2e^{-2\alpha}\gamma'' + 4\omega^2 = 2\Lambda + \kappa(1 + \omega_r + \omega_{\phi} + \omega_z)\rho, \quad (25)$$

$$2e^{-2\alpha}\mu'' = 2\Lambda + \kappa(-1 + \omega_r + \omega_{\phi} - \omega_z)\rho, \quad (26)$$

$$2e^{-2\alpha}\beta'' - 4\omega^2 = 2\Lambda + \kappa(-1 + \omega_r - \omega_{\phi} + \omega_z)\rho, \quad (27)$$

which are the t^t , z^z and ϕ^{ϕ} Eqs. (24), respectively. The ϕ^t Eq. (24) is a combination of (25) and (27) and the r^r Eq. (24) is a combination of (25), (26), (27) and $G_r^r = \omega^2 + e^{-2\alpha}(\beta'\gamma' + \beta'\mu' + \gamma'\mu')$ [recall that $G_r^r = T/2$].

5. Cosmological rotating wormholes

From now on, we assume $\omega_r \neq 0$ to ensure that (21) remains valid. We first construct cosmological rotating cylindrical wormholes to GR. The counterpart solutions to TEGR are the same with the torsion given by (22).

We look for solutions with a constant shift function, that is, $\gamma = 0$ and $\mu = 0$. In this case $\alpha = \beta$ (23) and Eq. (26) implies that ρ is a constant given by $\rho_0 = 2\Lambda/[\kappa(1 - \omega_r - \omega_{\phi} + \omega_z)]$. Since ρ is constant, Eq. (21) implies

$$\omega_{\phi} = \omega_r, \quad (28)$$

and finally

$$\rho_0 = \frac{2\Lambda}{\kappa(1 - 2\omega_r + \omega_z)}. \quad (29)$$

Eqs. (15) and (25) imply that ω^2 is also a constant proportional to the cosmological constant. This is given by

$$\omega_0^2 = \frac{\Lambda(1 + \omega_z)}{1 - 2\omega_r + \omega_z} = \frac{\kappa(1 + \omega_z)}{2} \rho_0. \quad (30)$$

Using these vales of (ρ_0, ω_0^2) in (27) we bring it to the form (recall $\alpha = \beta$)

$$e^{-2\alpha}\alpha'' = q^2 \equiv \frac{2\Lambda(1 - \omega_r + 2\omega_z)}{1 - 2\omega_r + \omega_z}. \quad (31)$$

This equation can be integrated in all three cases $q^2 < 0$, $q^2 = 0$ and $q^2 > 0$. In the case $q^2 < 0$ we obtain cosmological rotating solutions with two axes of symmetry at finite proper distance (see Section 7). The case $q^2 > 0$ (we may assume $q > 0$) yields cosmological rotating wormholes where the solution $e^{2\alpha}$ is brought to the form

$$e^{2\alpha} = e^{2\beta} = \frac{c^2 \sec^2(cr)}{q^2}, \quad (32)$$

with $c > 0$ being a constant of integration. As to Ω is obtained from (16)

$$\Omega(r) = 2\omega_0 \int^r e^{2\alpha(x)} dx = \frac{2c\omega_0}{q^2} \tan(cr), \quad (33)$$

where we have dropped an additive constant of integration. The metric takes the form

$$ds^2 = \left[dt - \frac{2c\omega_0 \tan(cr)}{q^2} d\phi \right]^2 - \frac{c^2 \sec^2(cr)}{q^2} dr^2 - \frac{c^2 \sec^2(cr)}{q^2} d\phi^2 - dz^2. \quad (34)$$

We see that the spherical radius $e^\beta = c \sec(cr)/q$ has a minimum value at $r = 0$ and increases as $|r|$ increases. On introducing the new radial coordinate $u = c \tan(cr)/q$ and the new constant $u_0^2 = c^2/q^2$, we bring the metric (34) to the manifestly wormhole form

$$ds^2 = \left(dt - \frac{2\omega_0 u}{q} d\phi \right)^2 - \frac{1}{q^2(u^2 + u_0^2)} du^2 - (u^2 + u_0^2) d\phi^2 - dz^2. \quad (35)$$

A very similar solution describing a rotating wormhole, which is a solution to the Einstein-Maxwell equations, was determined in [23].

TheTEGR wormhole solution is also given by (34) and (35) with $T = 2\omega_0^2 > 0$ (22), which is a positive constant.

The metric component

$$g_{\phi\phi} = -\left(\frac{q^2 - 4\omega_0^2}{q^2} u^2 + u_0^2 \right), \quad (36)$$

is manifestly negative if $q^2 - 4\omega_0^2 = 2\Lambda(1 + \omega_r)/(-1 + 2\omega_r - \omega_z) \geq 0$ signaling the absence of closed timelike curves (CTCs) [36]. For $q^2 - 4\omega_0^2 < 0$, CTCs occur at large values of $|u|$.

Let us see under which conditions the constraints $q^2 > 0$, $\omega_0^2 \geq 0$ and $\rho_0 > 0$ are satisfied simultaneously. The gravitational constant κ being positive, we obtain the following constraints on the EoS parameters $(\omega_r, \omega_\phi, \omega_z)$ assuming they obey the general inequalities (19). If $\Lambda < 0$ we obtain

$$\frac{1}{3} < \omega_r \leq 1 \text{ and } \frac{\omega_r - 1}{2} < \omega_z < 2\omega_r - 1. \quad (37)$$

If $\Lambda > 0$ we obtain

$$-1 \leq \omega_r \leq \frac{1}{3} \text{ and } \frac{\omega_r - 1}{2} < \omega_z \leq 1, \text{ or} \quad (38)$$

$$\frac{1}{3} < \omega_r < 1 \text{ and } 2\omega_r - 1 < \omega_z \leq 1. \quad (39)$$

If, however, we want to avoid the presence of CTCs, we have to impose the fourth constraint $q^2 - 4\omega_0^2 \geq 0$. This results in

$$\Lambda > 0, \quad \omega_r = -1 \text{ and } -1 < \omega_z \leq 1. \quad (40)$$

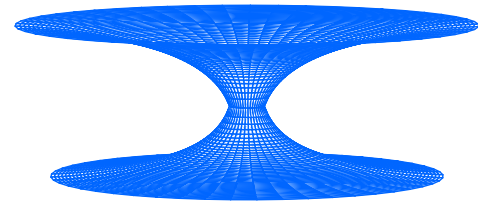


Fig. 1. The embedding diagram of the wormhole solution (35) for $\delta = 4/5$, $q = 3/5$ and $u_0 = 1/10$.

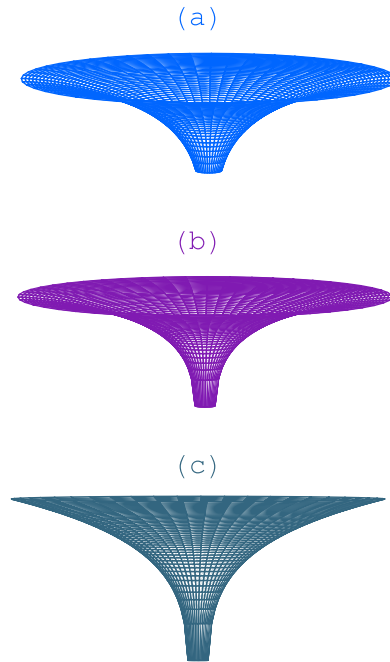


Fig. 2. The embedding diagram of the wormhole solution (35) showing the upper sheet only for $u_0 = 1/10$. (a) $\delta = 4/5$ and $q = 3/5$ (upper panel), (b) $\delta = 4/50$ and $q = 3/5$ (intermediate panel), (c) $\delta = 4/50$ and $q = 3/50$ (lower panel).

This shows that, for a positive cosmological constant, it is always possible to have a rotating wormhole with a positive energy density and no CTCs. The anisotropic fluid is isotropic in a plane perpendicular to the axis of rotation with $\omega_\phi = \omega_r = -1$.

To construct the embedding diagram of the metric (35) we introduce the radial coordinate R defined by $R^2 = \delta u^2 + u_0^2$ with $\delta \equiv (q^2 - 4\omega_0^2)/q^2$. For a constant time-slice and $z = \text{const.}$, the two-dimensional spatial metric takes the form

$$ds_2^2 = \frac{R^2 dR^2}{q^2(R^2 - u_0^2)[R^2 - (1 - \delta)u_0^2]} + R^2 d\phi^2.$$

This is to be embedded in the three-dimensional Euclidean space

$$ds_3^2 = dR^2 + R^2 d\phi^2 + dz^2.$$

The embedded diagram is a surface of revolution, symmetric with respect to the plane $Z = 0$, where $Z \equiv Z(R)$ with

$$Z(R) = \pm \int_{u_0\sqrt{1+\delta}}^R \sqrt{\frac{x^2}{q^2(x^2 - u_0^2)[x^2 - (1 - \delta)u_0^2]} - 1} dx.$$

Plots of the surface of revolution as depicted in Figs. 1 and 2. In Fig. 1 we have shown the two sheets of the wormhole for some set of the parameters (δ, q, u_0) while in Fig. 2 only the upper sheet is depicted for different values of the parameters (δ, q, u_0) .

6. Physical and geometrical properties of the rotating wormhole solution

Since the wormhole solutions (34) and (35) are not asymptotically flat, they are appropriate as matched interiors. Another reason why they are so is that the speed of rotation, $\Omega = 2\omega_0 u/q$ (33), increases linearly as one moves away from the axis of rotation at $u = 0$. Consequently the linear speed of each fluid particle increases unceasingly as one moves away from the axis of rotation. Following the work done in [23] one can match these wormholes to external rotating flat Minkowskian metrics in such a way that the radii of the cylindrical junction surfaces are chosen so that the interior wormhole region is exempt from CTCs.

Thus, the constraints (37)–(39), for $\Lambda < 0$ and $\Lambda > 0$, are fairly enough to obtain physically acceptable wormhole solutions that certainly do not violate the weak energy condition ($\rho_0 \geq 0$, $\rho_0 + p_r \geq 0$, $\rho_0 + p_z \geq 0$) and the dominant energy condition ($\rho_0 \geq 0$, $-\rho_0 \leq p_r \leq \rho_0$, $-\rho_0 \leq p_z \leq \rho_0$). To satisfy the requirements of the strong energy condition ($\rho_0 \geq 0$, $\rho_0 + p_r \geq 0$, $\rho_0 + p_z \geq 0$, $\rho_0 + 2p_r + p_z \geq 0$), the EoS parameters must obey the constraints (37) if $\Lambda < 0$ and the following constraints if $\Lambda > 0$:

$$\omega_r = -1 \text{ and } \omega_z = -1, \text{ or} \tag{41}$$

$$-1 < \omega_r < -\frac{1}{5} \text{ and } -2\omega_r - 1 \leq \omega_z \leq 1, \text{ or} \tag{42}$$

$$-\frac{1}{5} \leq \omega_r \leq \frac{1}{3} \text{ and } \frac{\omega_r - 1}{2} < \omega_z \leq 1, \text{ or} \tag{43}$$

$$\frac{1}{3} < \omega_r < 1 \text{ and } 2\omega_r - 1 < \omega_z \leq 1. \tag{44}$$

Considered as interiors all the matter should be distributed inside cylindrical surfaces Σ_+ and Σ_- of finite radii $u_+ > u_0$ and $u_- < -u_0$. Referring to [32,37], the effective mass m_+ and angular momentum j_+ per unit z-coordinate length of matter enclosed by Σ_+ are defined by (there are similar definitions for m_- and j_- concerning the matter distribution enclosed by Σ_-):

$$m_+ = 2\pi \int_0^{u_+} \tau_t^\mu n_\mu \sqrt{|g|} \, du, \tag{45}$$

$$j_+ = -2\pi \int_0^{u_+} \tau_\phi^\mu n_\mu \sqrt{|g|} \, du, \tag{46}$$

where τ_ν^μ is the total SET given in (13) and g is the determinant of the metric. The vector n_μ is the unit normal to the spacelike surface of integration, which is the hypersurface $t = \text{const.}$ yielding $n_\mu = \delta_\mu^t / \sqrt{g^{tt}}$ with g^{tt} being the component tt of the inverse metric. Using (35) along with $g = -1/q^2$ we find

$$m_+ = \frac{2\pi u_0}{q} \left(\rho_0 - \frac{\Lambda}{\kappa} \right) \int_0^{x_+} \sqrt{\frac{x^2 + 1}{\delta x^2 + 1}} \, dx, \tag{47}$$

$$j_+ = \frac{4\pi u_0^2 \rho_0 \omega_0 (1 + \omega_r)}{q^2} \int_0^{x_+} x \sqrt{\frac{x^2 + 1}{\delta x^2 + 1}} \, dx, \tag{48}$$

where $p_\phi = p_r = \omega_r \rho_0$ has been used, $x \equiv u/u_0$ and $x_+ \equiv u_+/u_0$. These integrals are expressed in terms of the incomplete elliptic integral of the second kind and the log function as given in the Appendix. It is obvious that j_+ has the sign of ω_0 and is zero in the static case (no rotation) and in the extreme case $\omega_r = -1$. The cosmological constant being small, we expect that in physically interesting situations to have $\rho_0 > |\Lambda|/\kappa$ and thus $m_+ > 0$.

The constants of integration (ω_0 , u_0/q) and $\rho_0 \propto \omega_0^2$ (30) could be determined analytically in terms of (m_+ , j_+ , x_+) if the expressions of (m_+ , j_+), as given in the Appendix, were invertible. This is, however, possible for $x_+ \ll 1$ where (m_+ , j_+) expand as (for all δ):

$$m_+ = \frac{\pi u_0 (\kappa \rho_0 - \Lambda) x_+ [6 + (1 - \delta) x_+^2]}{3q\kappa} + \mathcal{O}(x_+^5),$$

$$j_+ = \frac{\pi u_0^2 \rho_0 \omega_0 (1 + \omega_r) x_+^2 [4 + (1 - \delta) x_+^2]}{2q^2} + \mathcal{O}(x_+^6).$$

On eliminating the ratio u_0/q and using $\rho_0 = 2\omega_0^2/[\kappa(1 + \omega_z)]$ (30), we arrive at

$$9\kappa m_+^2 (1 + \omega_z) (1 + \omega_r) [4 + (1 - \delta) x_+^2] \omega_0^3 = \pi j_+ [6 + (1 - \delta) x_+^2] [\Lambda(1 + \omega_z) - 2\omega_0^2]^2,$$

which can be solved by radicals for ω_0 in terms of (m_+ , j_+ , x_+). The roots of this quartic equation in ω_0 , which can be obtained using a computer algebra system, are sizeable and cannot be reproduced in this work. Then, the ratio u_0/q is obtained from the above-given expansion of m_+ . Finally, using the expression of q (31) we can determine all the constants of integration in terms of the physical quantities (m_+ , j_+ , x_+ , ω_r , ω_z , Λ , κ).

The curvature scalar, \mathcal{R} , and the Kretschmann scalar, $R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}$, assume the following expressions:

$$\mathcal{R} = 2(\omega_0^2 - q^2), \tag{49}$$

$$R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} = 4(q^4 - 6\omega_0^2 q^2 + 11\omega_0^4), \tag{50}$$

which are finite constants.

7. Cosmological rotating solutions with two axes of symmetry at finite proper distance

If the constant q^2 in (31) is negative we set $Q^2 = -q^2$ in (31) and the solution yields

$$e^{2\alpha} = e^{2\beta} = \frac{c^2 \text{sech}^2(cr)}{Q^2}, \quad \Omega = \frac{2c\omega_0}{Q^2} \tanh(cr), \tag{51}$$

where we have dropped an additive constant Ω_0 in the expression of Ω . On setting $u = c \tanh(cr)/Q$ and $u_0^2 = c^2/Q^2$, we bring the metric to the form

$$ds^2 = \left[dt - \frac{2c\omega_0 \tanh(cr)}{Q^2} d\phi \right]^2 - \frac{c^2 \text{sech}^2(cr)}{Q^2} dr^2 - \frac{c^2 \text{sech}^2(cr)}{Q^2} d\phi^2 - dz^2, \tag{52}$$

$$ds^2 = \left(dt - \frac{2\omega_0 u}{Q} d\phi \right)^2 - \frac{1}{Q^2(u_0^2 - u^2)} du^2 - (u_0^2 - u^2) d\phi^2 - dz^2. \tag{53}$$

Note that this solution shares with the wormhole solution (34)–(35) the same values of the physical quantities (ρ_0 , ω_0^2), given in (29) and (30). The correspondingTEGR solution is also given by (52)–(53) and shares the same value of the torsion $T = 2\omega_0^2$ with the wormhole solution (34)–(35).

The circular radius $e^\beta = \sqrt{u_0^2 - u^2}$ vanishes at $u = \pm u_0$ signaling the presence of two axes of symmetry. In the vicinity of these two axes the solution (52)–(53) has CTCs where $g_{\phi\phi} = -(u_0^2 - u^2 - 4\omega_0^2 u^2/Q^2)$ becomes positive (the nonrotating solution, $\omega_0 = 0$, has no CTCs). The integral

$$\int_{-u_0}^{u_0} \frac{1}{Q\sqrt{u_0^2 - u^2}} \, du = \frac{\pi}{Q},$$

being convergent, the two axes are at finite proper distance from each other and there is no spatial infinity. The absence of spatial infinity is also known for the nonrotating Melvin solution [36,38,39], however, for the latter the proper distance of the two axes

of symmetry

$$\int_0^\infty (1 + B^2 u^2) du,$$

diverges (B is the magnetic field). Such nontrivial behaviors of the intrinsic geometry are familiar with static and rotating, cylindrically symmetric and/or axially symmetric, metrics and more examples are provided in [36].

8. Conclusion

As we mentioned in the Introduction, the determination of rotating and static solutions around an infinite axis is still attracting much attention. In this work, we presented a first set of two cosmological (energy density constant) rotating solutions in GR andTEGR gravity sourced by anisotropic fluids (isotropic in a plane perpendicular to the axis of symmetry) and extended the existing list of solutions pertaining to GR. These solutions have the property that their angular velocity is proportional to the cosmological constant.

We have shown that the EoS parameters obey a large set of values ensuring the satisfaction of all local energy conditions for the rotating wormholes. Such solutions can straightforwardly be matched to exterior rotating Minkowskian metrics.

The other cosmological rotating solution has two axes of symmetry at finite proper distance where one axis can be regularized upon appropriately fixing the value of the additive constant of integration in the expression of Ω (which we have dropped) at the expense of rendering the time coordinate periodic.

CRediT authorship contribution statement

Mustapha Azreg-Ainou: Conceptualization, Calculation, Plotting graphs, Writing - original draft, Visualization, Reviewing and editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Analytical expression for (m_+, j_+)

The analytical expressions of (m_+, j_+) depend on the sign of δ , however, their series expansions about $x_+ = 0$ do not, as this is shown in the new paragraph following (48). For $\delta > 0$ we have

$$m_+ = -\frac{2i\pi u_0(\kappa\rho_0 - \Lambda)}{\kappa q\sqrt{\delta}} E\left(i \ln(\sqrt{\delta}x_+ + \sqrt{1 + \delta x_+^2}) \middle| \frac{1}{\delta}\right),$$

$$j_+ = \frac{2\pi u_0^2 \rho_0 \omega_0 (1 + \omega_r)}{\delta^{3/2} q^2} \left[\sqrt{1 + \delta x_+^2} \sqrt{\delta(1 + x_+^2)} - \sqrt{\delta} + (1 - \delta) \ln\left(\frac{1 + \sqrt{\delta}}{\sqrt{1 + \delta x_+^2} + \sqrt{\delta(1 + x_+^2)}}\right) \right].$$

For $\delta = 0$, we have

$$m_+ = \frac{\pi u_0(\kappa\rho_0 - \Lambda)[x_+ \sqrt{1 + x_+^2} + \ln(x_+ + \sqrt{1 + x_+^2})]}{\kappa q},$$

$$j_+ = \frac{4\pi u_0^2 \rho_0 \omega_0 (1 + \omega_r) [(1 + x_+^2) \sqrt{1 + x_+^2} - 1]}{3q^2}.$$

For $\delta < 0$ we obtain

$$m_+ = \frac{2\pi u_0(\kappa\rho_0 - \Lambda)}{\kappa q\sqrt{-\delta}} E\left(-i \ln(i\sqrt{-\delta}x_+ + \sqrt{1 + \delta x_+^2}) \middle| \frac{1}{\delta}\right),$$

$$j_+ = \frac{2\pi u_0^2 \rho_0 \omega_0 (1 + \omega_r)}{\delta^{3/2} q^2} \left[i\sqrt{1 + \delta x_+^2} \sqrt{-\delta(1 + x_+^2)} - i\sqrt{\delta} + (1 - \delta) \ln\left(\frac{1 + i\sqrt{-\delta}}{\sqrt{1 + \delta x_+^2} + i\sqrt{-\delta(1 + x_+^2)}}\right) \right],$$

where $0 < x_+ < 1/\sqrt{-\delta}$ if $\delta < 0$.

In all these expressions $i^2 = -1$ and

$$E(z|m) \equiv \int_0^z \sqrt{1 - m \sin^2 \theta} d\theta,$$

is the incomplete elliptic integral of the second kind. Here z is generally a complex number input. Despite a complex input, the output of the above-given expressions of (m_+, j_+) is always real for all δ (for $\delta < 0$ the output is real provided $0 < x_+ < 1/\sqrt{-\delta}$).

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